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Solving open string field theory with special projectors

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Abstract: Schnabl recently found an analytic expression for the string field tachyon condensate using a gauge condition adapted to the conformal frame of the sliver projector. We propose that this construction is more general. The sliver is an example of a *special* projector, a projector such that the Virasoro operator $\mathcal{L}_0$ and its BPZ adjoint $\mathcal{L}_0^\star$ obey the algebra $[\mathcal{L}_0, \mathcal{L}_0^\star] = s(\mathcal{L}_0 + \mathcal{L}_0^\star)$, with s a positive real constant. All special projectors provide abelian subalgebras of string fields, closed under both the $*$-product and the action of $\mathcal{L}_0$. This structure guarantees exact solvability of a ghost number zero string field equation. We recast this infinite recursive set of equations as an ordinary differential equation that is easily solved. The classification of special projectors is reduced to a version of the Riemann-Hilbert problem, with piecewise constant data on the boundary of a disk.

Keywords: String Field Theory, Bosonic Strings, D-branes, Tachyon Condensation.

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1. Introduction and summary

The classical equation of motion of open string field theory,

$$Q_B\Psi + \Psi * \Psi = 0\,, \tag{1.1}$$

is a notoriously complicated system of infinitely many coupled equations. In a recent breakthrough, Schnabl constructed the first analytic solution of (1.1), the string field that represents the stable vacuum of the open string tachyon [1]. The solution has been subject to important consistency checks [2, 3], it has been presented in alternative forms [2], and it has been used very recently to discuss the absence of physical states at the tachyon vacuum [4].

In the resurgence of string field theory triggered by Sen's conjectures on tachyon condensation [5], many new techniques were developed to deal more efficiently with the open string star product [6], e.g. [7 − 26], see [27 − 29] for recent reviews. One line of development [11] emphasized the special role of "projector" string fields, i.e. string fields squaring to themselves,

$$\Phi * \Phi = \Phi\,. \tag{1.2}$$

Particularly simple are *surface state* projectors [7, 10, 17]. In the operator formalism of conformal field theory, the surface state $\langle f|$ is the state associated with the one-punctured disk whose local coordinate around the puncture is specified by the conformal map $z = f(\xi)$ from a canonical half-disk to the z upper-half plane.[1] If $f(i) = \infty$, the open string midpoint $\xi = i$ is mapped to the (conformal) boundary of the disk, and the corresponding surface state $\langle f|$ is a projector. The sliver [7 − 10] and the butterfly [17] are the two most studied examples of this construction. Additional structure is known for the sliver. There is a continuous family of wedge states W_α, with $\alpha \geq 0$,[2] that interpolate between the identity string field $W_0 \equiv \mathcal{I}$ and the sliver W_∞ [7, 10]. The wedge surface states obey the simple abelian multiplication rule

$$W_\alpha * W_\beta = W_{\alpha+\beta}\,, \quad \alpha\,, \beta \in [0,\ \infty)\,. \tag{1.3}$$

It seems natural to look for an analytic solution of (1.1) that makes use of this abelian family. We may look for an expansion of Ψ in terms of wedge states, with extra ghost insertions necessary to obtain ghost number one string fields. However, early attempts to solve the equations of motion in the Siegel gauge $b_0\Psi = 0$,

$$L_0\Psi + b_0(\Psi * \Psi) = 0\,, \tag{1.4}$$

[1] The function $z = f(\xi)$ is analytic in the half-disk $H_U = \{|\xi| < 1\,,\ \Im\xi \geq 0\}$, which is mapped to a neighborhood of $z = 0$, with $f(0) = 0$. The real boundary of H_U is mapped inside the boundary of the UHP.

[2] $W_\alpha = |\alpha + 1\rangle$ in the notation of [7].

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were frustrated by the predicament that while the star product is simple in the wedge basis, the action of L_0 is not.[3] Schnabl's main observation [1] is to choose instead a gauge $\mathcal{B}_0\Psi = 0$ well adapted to the wedge subalgebra. Here $\mathcal{B}_0$ is the antighost zero mode in the conformal frame $z = f(\xi) = \arctan(\xi)$ of the sliver:

$$\begin{aligned}\mathcal{B}_0 &\equiv \oint \frac{dz}{2\pi i}\, z\, b(z) = \oint \frac{d\xi}{2\pi i}\frac{f(\xi)}{f'(\xi)}\, b(\xi)\\ &= \oint \frac{d\xi}{2\pi i}(1+\xi^2)\tan^{-1}(\xi)\, b(\xi) = b_0 + \frac{2}{3}\, b_2 - \frac{2}{15}\, b_4 + \ldots\end{aligned} \tag{1.5}$$

The corresponding kinetic operator is the stress tensor zero-mode in the conformal frame of the sliver:

$$\begin{aligned}\mathcal{L}_0 &\equiv \oint \frac{dz}{2\pi i}\, z\, T(z) = \oint \frac{d\xi}{2\pi i}\frac{f(\xi)}{f'(\xi)} T(\xi)\\ &= \oint \frac{d\xi}{2\pi i}(1+\xi^2)\tan^{-1}(\xi)\, T(\xi) = L_0 + \frac{2}{3}\, L_2 - \frac{2}{15}\, L_4 + \ldots\,.\end{aligned} \tag{1.6}$$

Here $T(z)$ is the total stress tensor, which has zero central charge and is a true conformal primary of dimension two. The operator $\mathcal{L}_0$ has a simple action on the W_α states. In this gauge, the equation of motion becomes exactly solvable in terms of wedge states (with ghost insertions)!

In this paper we ask how general is the strategy of Schnabl and investigate its algebraic structure. What are the algebraic properties of the sliver gauge that ensure solvability? Why is a projector relevant? For simplicity, we shall actually focus on the equation of motion [1, 31]

$$(\mathcal{L}_0 - 1)\Phi + \Phi * \Phi = 0\,, \tag{1.7}$$

where Φ has ghost number zero. This equation captures many important features of the full equation of motion (1.1). We are optimistic that the algebraic structures discussed in this context will generalize to (1.1). In equation (1.7), $\mathcal{L}_0$ is the zero mode of the stress tensor in some generic conformal frame $z = f(\xi)$. We find an infinite class of local coordinate maps $f(\xi)$ that lead to solvable equations of the form (1.7) – all the maps obeying the three following conditions:

1. The operator $\mathcal{L}_0$ and its BPZ conjugate $\mathcal{L}_0^\star$ satisfy the algebra

   $$[\,\mathcal{L}_0\,, \mathcal{L}_0^\star\,] = s\,(\mathcal{L}_0 + \mathcal{L}_0^\star)\,, \tag{1.8}$$

   where s is a positive real number.

2. The BPZ even operator $L^+ \equiv \frac{1}{s}(\mathcal{L}_0 + \mathcal{L}_0^\star)$ admits a non-anomalous left/right decomposition $L^+ = L_L^+ + L_R^+$.

3. The BPZ odd operators $L^- \equiv \frac{1}{s}(\mathcal{L}_0 - \mathcal{L}_0^\star)$ and $K \equiv L_R^+ - L_L^+$ annihilate the identity string field $\mathcal{I}$ of the $*$-algebra.

[3] A closely related difficulty was encountered in the Moyal formulation of the open string star product [18, 19]. This question has now been reconsidered in light of the new developments [30].

Condition 1 is a strong restriction on conformal frames. Conditions 2 and 3, described in detail in section 2, can be interpreted as regularity conditions for the vector $v(\xi)$ associated[4] with $\mathcal{L}_0$. In particular, $v(\xi)$ must vanish at the string midpoint.

Our analysis of the above conditions for solvability leads to a nontrivial result: the functions $f(\xi)$ that satisfy all of them define projectors! We find this quite interesting, for it explains the relevance of projectors to solvability. Not all projectors satisfy all three conditions. The projectors for which conditions 1,2, and 3 are satisfied will be called *special projectors* and will be the focus of this paper. In summary, the conformal frames f associated with special projectors lead to solvable equations.

It is quite easy to find concrete maps that define special projectors. The sliver, of course, is the original example. The operator $\mathcal{L}_0$ in (1.6) satisfies (1.8) with $s = 1$ [16, 1]. But the simplest example is actually the butterfly,

$$f(\xi) = \frac{\xi}{\sqrt{1+\xi^2}} \quad \rightarrow \quad \mathcal{L}_0 = L_0 + L_2\,, \quad \mathcal{L}_0^\star = L_0 + L_{-2}\,. \tag{1.9}$$

An elementary calculation reveals that the butterfly satisfies (1.8) with $s = 2$.

In section 2, we show that for each special projector $\langle f|$, there exists an abelian subalgebra $\mathcal{A}_f$, closed both under $*$-product and the action of the kinetic operator $\mathcal{L}_0$. The subalgebra is constructed as

$$\mathcal{A}_f = \mathrm{Span}(\chi_n)\,, \quad \chi_n \equiv (L^+)^n\,\mathcal{I}\,, \quad n \geq 0\,. \tag{1.10}$$

The basis states obey

$$\chi_m * \chi_n = \chi_{m+n}\,, \tag{1.11}$$

as well as

$$\mathcal{L}_0\,\chi_n = s\left(n\chi_n + \frac{1}{2}\chi_{n+1}\right)\,. \tag{1.12}$$

Our proof of these properties will be algebraic. An essential point will be the decomposition of operators into a "left" and "right" part, acting on the left and right halves of the open string.

Interesting elements of $\mathcal{A}_f$ are the surface states P_α, studied in section 3:

$$P_\alpha \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\alpha}{2}\right)^n \chi_n = \exp\left(-\frac{\alpha}{2}L^+\right)\mathcal{I}\,, \quad \alpha \geq 0\,. \tag{1.13}$$

From (1.11),

$$P_\alpha * P_\beta = P_{\alpha+\beta}\,. \tag{1.14}$$

Using the sliver's L^+, this construction gives the familiar wedge states: $P_\alpha \equiv W_\alpha$. For surface states $\langle f|$ that satisfy conditions 2 and 3 we have states P_α that satisfy (1.14). The state P_∞, if it exists, is then a projector. If $\langle f|$ also satisfies condition 1, we can prove that P_∞ coincides with $\langle f|$. Thus we learn that $\langle f|$ is a projector, in fact, a special projector

[4] To a vector field $v(\xi)$ one naturally associates the linear combinations of Virasoro operators given by the conformally invariant integral $\int \frac{d\xi}{2\pi i} T(\xi) v(\xi)$.

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according to our definition. In summary, for every special projector $\langle f|$ the P_α obey the abelian multiplication rule (1.14) and interpolate between the identity and $\langle f|$ itself. In constructing the above argument we have learned of a general property of the operator $\mathcal{L}_0$ associated with an arbitrary conformal frame $f(\xi)$: the corresponding surface state $\langle f|$ can be written as

$$\langle f| = \lim_{\gamma\to\infty} \langle\Sigma|\, e^{-\gamma\mathcal{L}_0}\,, \tag{1.15}$$

where $\langle\Sigma|$ is an arbitrary surface state. While any (bra) surface state is annihilated by its corresponding $\mathcal{L}_0$, special projectors are also annihilated by $\mathcal{L}_0^\star$.

Properties (1.11) and (1.12) guarantee that we can consistently solve the ghost number zero equation (1.7) with $\Phi \in \mathcal{A}_f$. As described in section 4, this is most efficiently done by considering the ansatz

$$\Phi = f_s(x)\,\mathcal{I}\,, \qquad x \equiv L^+\,. \tag{1.16}$$

The equation of motion (1.7) is translated to a differential equation for $f_s(x)$, which is easily solved for all values of s. For the familiar case of the sliver ($s = 1$), Schnabl's solution of (1.7) was obtained by recognizing the appearance of Bernoulli numbers in an infinite set of recursive equations [1]. For us, the solution of the $s = 1$ differential equation is the generating function of Bernoulli numbers. The solution for $s = 2$ is given in terms of the error function and for general s in terms of hypergeometric functions. The higher s solutions provide, in some sense, an s-dependent deformation of the Bernoulli numbers, which arise from the $s = 1$ solution.

Equivalently, in a recursive analysis one can focus on the Taylor expansion of the function $f_s(x)$ in (1.16):

$$\Phi = f_s(x)\,\mathcal{I} = \sum_{n=0}^{\infty} a_n^{(s)}\, x^n\, \mathcal{I} = \sum_{n=0}^{\infty} a_n^{(s)} \chi_n\,. \tag{1.17}$$

We define a level ℓ^+ that counts the powers of L^+ acting on the identity. This assigns level n to the basis states χ_n. The level ℓ^+ is exactly additive under $*$-product: the product of two string fields Φ_1 and Φ_2 of definite level satisfies

$$\ell^+(\Phi_1 * \Phi_2) = \ell^+(\Phi_1) + \ell^+(\Phi_2)\,. \tag{1.18}$$

Because of (1.12), $\mathcal{L}_0$ acting on a term of definite level gives terms with the same level and terms with level increased by one. Thus we can set up solvable recursion relations for the coefficients $a_n^{(s)}$: the term in the equation of motion proportional to χ_N depends only on $a_n^{(s)}$ with $n \le N$ so that $a_N^{(s)}$ can be determined. A level is said to be super(sub)-additive under a product if the product of two factors of definite level produces terms with levels greater (less) than or equal to the sum of levels of the factors. If we have a level that is super-additive under $*$-product and does not decrease by $\mathcal{L}_0$ action, we still have an exactly solvable recursion. The ansatz in [1], with basis states $x^n P_1$ of level n, is super-additive under $*$-product. In fact, the product of two such basis states produces states of arbitrarily higher level. Our ansatz (1.16) leads to an even simpler recursion, since in our basis the product is *exactly* additive.

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The exact solution can also be presented as a linear superposition of P_α states acted by L^+:

$$\Phi = P_\infty + \int_0^\infty d\alpha\, \mu_s(\alpha)\, L^+\, P_\alpha\,. \tag{1.19}$$

The density function $\mu_s(\alpha)$ turns out to be the inverse Laplace transform of $\frac{f_s(2x)}{2x}$. For $s = 1$ we reproduce the expression obtained in [1]: $\mu_1(\alpha)$ is a sum of delta functions localized at $\alpha = n$ for $n = 1, 2, 3, \ldots \infty$. For $s > 1$ one has instead a continuous superposition of P_α states with $\alpha \geq 1$, namely,

$$\mu_s(\alpha) = 0 \quad \text{for} \quad \alpha < 1\,. \tag{1.20}$$

The states P_α with $\alpha < 1$ and, in particular, the identity string field $P_0 = \mathcal{I}$ are absent in this form of the solution. This may be important since formal solutions of string field theory directly based on the identity (e.g. [32 − 34]) have turned out to be singular.

Having found the exact solution, we show how to re-write it in a better ordered form,

$$\Phi = g_s(u)\mathcal{I}\,, \quad u = L^\star\,. \tag{1.21}$$

The function g_s is obtained from f_s as a linear integral transform closely related to the Mellin transform. As opposed to L^+, the operator $L^\star$ does not contain positively moded Virasoro operators. From here it is a short step to obtain the solution in the ordinary level expansion, that is, as a linear combination of Virasoro descendants of the $\mathrm{SL}(2, R)$ vacuum.

The form (1.21), expanded in powers of u, does not give an exactly solvable recursion. Indeed, the level $\ell^\star$, defined by $\ell^\star(u^n\mathcal{I}) = n$, is sub-additive under the $*$-product and the action of $\mathcal{L}_0$ both increases and decreases the level. Nevertheless, a level truncation approximation scheme is easy to set-up in this basis because the products take a rather simple form. In section 5 we show that $\ell^\star$-level truncation converges very rapidly to the exact answer. Thus $\ell^\star$-level truncation provides a new approximation scheme for string theory. It is interesting to compare with the extensively used ordinary level truncation [35 − 38], where the level ℓ is defined to be the eigenvalue of L_0+1. Ordinary level truncation is less tractable than $\ell^\star$-level truncation. While the kinetic operator in the Siegel gauge preserves ordinary level (by definition!) the star product of two fields with definite levels generally produces states of all levels. Moreover, the coefficients of the products are complicated to evaluate.

All the preceding analysis does not assume any specific form for the map $f(\xi)$ that defines the special projector. In section 6 we look at some concrete examples of projectors. In section 6.1 we consider a one-parameter family of projectors and highlight the distinguishing features that emerge when the parameter is tuned so that the projector is special. In section 6.2 we illustrate our general framework studying in some detail the example of the butterfly and some of its generalizations. We construct explicitly the family of states P_α that interpolate between the identity and the butterfly. For large α these states provide an exact regulator of the butterfly; previously found regulators of the butterfly state only closed approximately under $*$-multiplication.

In section 7, we pose the problem of finding the most general special conformal frame — a frame that leads to the algebra (1.8), irrespective of whether it also satisfies conditions 2 and 3. Surprisingly, an analysis that begins with a second-order differential equation for

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$f(\xi)$ turns out to give a linear constraint equation that involves only the values of $f(\xi)$ and its complex conjugate on the circle $|\xi| = 1$. This constraint in fact implies that $f(\xi)$ is the analytic function that solves the classic Riemann-Hilbert problem on the interior of a disk for the case of piecewise constant data on the circle boundary. It is pleasant to find that the question of the algebra of $\mathcal{L}_0$ and $\mathcal{L}_0^\star$ leads to such natural mathematical problem. We do not perform an exhaustive analysis, but our results so far suggest that the sliver is the unique special projector with $s = 1$, and that for each integer s there is only a finite number of special projectors. For $s = 2$ we discuss two special projectors, the butterfly and the moth. Our results suggest a role for the Virasoro operator $\mathcal{L}_{-s}$ in the conformal frame of the projector. We offer some concluding remarks in section 8. An appendix collects some useful algebraic identities.

2. The abelian subalgebra

In this section we introduce and study the abelian subalgebra $\mathcal{A}_f$. We begin by defining the operator $\mathcal{L}_0$, which can be viewed as the L_0 operator in the conformal frame $z = f(\xi)$:

$$\mathcal{L}_0 = \oint \frac{dz}{2\pi i}\, z\, T(z) = \oint \frac{d\xi}{2\pi i}\, \frac{f(\xi)}{f'(\xi)}\, T(\xi)\,. \tag{2.1}$$

The BPZ dual of $\mathcal{L}_0$ is denoted by $\mathcal{L}_0^\star$.[5] A number of formal algebraic properties will be assumed. The first property is that these operators obey the algebra

$$[\underline{\mathbf{1}}] \qquad\qquad [\, \mathcal{L}_0\,, \mathcal{L}_0^\star\,] = s(\mathcal{L}_0 + \mathcal{L}_0^\star)\,, \quad s > 0\,.$$

We define normalized operators

$$L \equiv \frac{1}{s}\mathcal{L}_0 \quad \text{and} \quad L^\star \equiv \frac{1}{s}\mathcal{L}_0^\star\,, \tag{2.2}$$

so that the algebra takes the canonical form

$$[\underline{\mathbf{1}}] \qquad\qquad [\, L\,, L^\star\,] = L + L^\star\,.$$

Next, we form a BPZ even combination L^+ and a BPZ odd combination L^-:

$$L^+ \equiv L + L^\star\,, \qquad L^- \equiv L - L^\star\,. \tag{2.3}$$

We shall define a precise notion of the *left* and *right* part of an operator, acting on the left and the right half of the open string. The BPZ even operator L^+ is split into a left and a right part

$$L^+ = L^+_L + L^+_R\,. \tag{2.4}$$

[5] In [1] the operator $\mathcal{L}_0^\star$ is written as the hermitian conjugate $\mathcal{L}_0^\dagger$ of $\mathcal{L}_0$. We use the $\star$ because, in all generality, the algebraic framework requires the use of BPZ conjugation. For the sliver, BPZ and hermitian conjugation agree, as they do for all twist even projectors. If one were to deal with non twist even projectors, one must use BPZ conjugation.

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We demand that formal properties of left/right splitting actually hold (i.e. are not anomalous). Concretely, we require

$$[\,\underline{\mathbf{2a}}\,] \qquad [\,L_L^+\,,\,L_R^+\,] = 0\,, \tag{2.5}$$

$$[\,\underline{\mathbf{2b}}\,] \qquad L_L^+(\Phi_1 * \Phi_2) = (L_L^+\Phi_1) * \Phi_2\,. \tag{2.6}$$

Conditions [**2a**] and [**2b**] describe precisely condition 2, as stated in the introduction.

Given an operator $\mathcal{O} = \mathcal{O}_L + \mathcal{O}_R$, we define its *dual* $\widetilde{\mathcal{O}}$ as

$$\widetilde{\mathcal{O}} \equiv \mathcal{O}_R - \mathcal{O}_L\,. \tag{2.7}$$

The duals of L^+ and L^- are denoted by K and J, respectively,

$$\begin{aligned} K &\equiv \widetilde{L^+} \equiv L_R^+ - L_L^+\,,\\ J &\equiv \widetilde{L^-} \equiv L_R^- - L_L^-\,. \end{aligned} \tag{2.8}$$

Duality, as we will demonstrate, reverses BPZ parity, so K is BPZ odd and J is BPZ even. The operators L^- and K, being BPZ odd, are naively expected to be derivations of the $*$-algebra and to annihilate the identity. As we shall see, these properties can be anomalous and must be checked explicitly, so we highlight them as our last formal properties,

$$[\,\underline{\mathbf{3a}}\,] \qquad L^-\mathcal{I} = (L - L^\star)\,\mathcal{I} = 0\,, \tag{2.9}$$

$$[\,\underline{\mathbf{3b}}\,] \qquad K\,\mathcal{I} = (L_R^+ - L_L^+)\,\mathcal{I} = 0\,. \tag{2.10}$$

Conditions [**3a**] and [**3b**] describe precisely condition 3, as stated in the introduction.

Finally, we define the subspace $\mathcal{A}_f$ as

$$\mathcal{A}_f = \mathrm{Span}(\chi_n)\,, \quad \chi_n \equiv (L^+)^n\,\mathcal{I} \quad n \geq 0\,. \tag{2.11}$$

We claim that if all the formal properties hold, $\mathcal{A}_f$ is actually a subalgebra, closed under both the $*$-product and the action of L. Let us demonstrate this explicitly:

- *Closure under L.* We need to assume only [**1**] and [**3a**]. Indeed,

$$L\,\chi_n = L\,(L^+)^n\mathcal{I} = [L,\,(L^+)^n]\,\mathcal{I} + (L^+)^n L\mathcal{I}\,. \tag{2.12}$$

  Using [**1**] to compute the commutator in the first term and [**3a**] to re-write the second term,

$$L\chi_n = n(L^+)^n\mathcal{I} + \frac{1}{2}(L^+)^{n+1}\mathcal{I} = n\chi_n + \frac{1}{2}\,\chi_{n+1}\,, \tag{2.13}$$

  as claimed.

- *Closure under $*$.* We need to assume only [**2a**], [**2b**], and [**3b**]. Indeed, using [**3b**] and [**2a**] we can write

$$\chi_n = (L^+)^n\,\mathcal{I} = (L^+)^{n-1}\,(2L_L^+)\,\mathcal{I} = (L^+)^{n-k}\,(2L_L^+)^k\,\mathcal{I} = (2L_L^+)^k\,(L^+)^{n-k}\,\mathcal{I} \tag{2.14}$$

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for any integer k, $0 \le k \le n$. Then, from repeated application of [**2b**],

$$\chi_m * \chi_n = (2L_L^+)^m |\mathcal{I}\rangle * \chi_n = (2L_L^+)^m \, (\mathcal{I} * \chi_n) = (2L_L^+)^m \, \chi_n \,, \qquad (2.15)$$

which by (2.14) is recognized as χ_{m+n}. In summary

$$\chi_m * \chi_n = \chi_{m+n} \,. \qquad (2.16)$$

We can give a simple explanation of the multiplication rule (2.16). Because of [**3a**] and [**1**], we see that the basis states χ_n are eigenstates of L^-:

$$\tfrac{1}{2} L^- \, \chi_n = \tfrac{1}{2} [L^-, (L^+)^n] \, \mathcal{I} = n \, \chi_n \,. \qquad (2.17)$$

The eigenvalue of $\frac{1}{2}L^-$ will be called the level ℓ^+, so we have $\ell^+(\chi_n) = n$. Since L^- is a derivation, it follows that ℓ^+ is additive under the $*$-product, thus explaining (2.16). By contrast, the level in [1] was defined as the eigenvalue of L. The eigenstates of L are $(L^+)^n|P_1\rangle$ because, as we shall show in section 3.2, $L|P_1\rangle = 0$. Finally, since L is not a derivation this level is not additive — it is in fact super-additive.

2.1 Vector fields and BPZ conjugation

To any vector field $v(\xi)$ we associate the stress-energy insertion $\mathbf{T}(v)$ defined by

$$\mathbf{T}(v) \equiv \oint \frac{d\xi}{2\pi i} \, v(\xi) \, T(\xi) \,, \qquad (2.18)$$

where the integral is performed over some contour that encircles the origin $\xi = 0$. The choice of contour can be important if $v(\xi)$ has singularities. In this notation, (2.1) is written as

$$\mathcal{L}_0 = \mathbf{T}(v) \,, \qquad v(\xi) = \frac{f(\xi)}{f'(\xi)} \,. \qquad (2.19)$$

In this case the vector v does not have singularities for $|\xi| < 1$, so the closed contour can be taken in this domain. A general identity that follows from the definition (2.18) and the OPE of two stress tensors (with zero central charge) is

$$[\, \mathbf{T}(v_1) \,, \, \mathbf{T}(v_2) \,] = -\mathbf{T}(\, [v_1, v_2] \,) \,, \qquad [v_1, v_2] \equiv v_1 \partial v_2 - v_2 \partial v_1 \,. \qquad (2.20)$$

We will denote by $(\mathbf{T}(v))^\star$ the BPZ conjugate of $\mathbf{T}(v)$. Recall that under BPZ conjugation $L_n \to (-)^n L_{-n}$. For a vector field $v(\xi) = \sum v_n \, \xi^{n+1}$ we find

$$\mathbf{T}(v) = \sum_n v_n L_n \quad \to \quad (\mathbf{T}(v))^\star = \sum_n v_n (-1)^n L_{-n} = \sum_n v_{-n} (-1)^n L_n \,. \qquad (2.21)$$

We thus find that

$$(\mathbf{T}(v))^\star = \mathbf{T}(v^\star) \,, \qquad (2.22)$$

where the BPZ conjugate vector $v^\star$ is

$$v^\star(\xi) = \sum_n v_{-n} (-1)^n \, \xi^{n+1} = -\, \xi^2 \sum_n v_n \, \frac{1}{(-\xi)^{n+1}} \,. \qquad (2.23)$$

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We thus recognize that

$$v^\star(\xi) = -\xi^2\, v(-1/\xi)\,. \tag{2.24}$$

Generally, the vector $v(\xi)$ is analytic in the interior $|\xi| < 1$ of the unit disk, with possible singularities on the circle $|\xi| = 1$. It then follows that the BPZ conjugate vector $v^\star$ is analytic outside of the unit disk with possible singularities on the unit circle.

Vectors of definite BPZ parity must be considered with some care. For example, given a vector v we can construct the vector $v^\star$ and then form the vectors

$$v^\pm = v \pm v^\star \tag{2.25}$$

The vector v^+ is said to be BPZ even and the vector v^- is said to be BPZ odd. The domain of definition of the vectors $v^\pm$ is the common domain of analyticity of v and $v^\star$. This domain of $v^\pm$ is the whole plane for the vector $v = \alpha + \beta\xi + \gamma\xi^2$, with arbitrary constants α, β, and γ. The domain of $v^\pm$ is an annulus around the circle $|\xi| = 1$ for e.g. the vector $v = \xi^3$, since v is singular at $\xi = \infty$ while $v^\star$ is singular for $\xi = 0$. When v has branch cuts that emerge from points on the circle $|\xi| = 1$, we use the circle (minus the singular points) as the domain of $v^\pm$. A BPZ even (odd) vector leads to a BPZ even (odd) operator $\mathbf{T}$.

To discuss the symmetry properties of BPZ even/odd operators we use $t = e^{i\theta}$ for the points on the circle $|\xi| = 1$. Since $1/t = t^*$ (here $*$ is complex conjugation), we have that (2.24) gives

$$v^\star(t) = -\frac{1}{(t^*)^2}\, v(-t^*)\,. \tag{2.26}$$

It then follows from (2.25) that

$$\frac{v^\pm(t)}{t} = \frac{v(t)}{t} \pm \frac{v(-t^*)}{(-t^*)}\,. \tag{2.27}$$

Since $t \to -t^*$ is a reflection about the imaginary axis, we learn that v^+/t is invariant under reflection about the imaginary axis while v^-/t changes sign under this reflection. If additionally, $v(-t) = -v(t)$ and $v(t^*) = (v(t))^*$, then

$$\frac{v^\pm(t)}{t} = \frac{v(t)}{t} \pm \left(\frac{v(t)}{(t)}\right)^*\,. \tag{2.28}$$

In that case we see that v^+/t is real while v^-/t is imaginary.

2.2 Left/right splitting and duality

Consider (2.18) in which the integral will be performed over the unit circle C defined by $|\xi| = 1$. For clarity, we continue to use t to represent points on the $|\xi| = 1$ circle. We write

$$\mathbf{T}(v) \equiv \oint_C \frac{dt}{2\pi i}\, v(t)\, T(t)\,, \tag{2.29}$$

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We call the part of the circle with Re $t > 0$ the right part C_R and the part of the circle with Re $t < 0$ the left part C_L. Associated with a vector $v(t)$ — in general only defined on the unit circle — we introduce the left part v_L and the right part v_R:

$$v_L(t) = \begin{cases} v(t)\,, & \text{if } t \in C_L; \\ 0\,, & \text{if } t \in C_R; \end{cases} \qquad v_R(t) = \begin{cases} 0\,, & \text{if } t \in C_L; \\ v(t)\,, & \text{if } t \in C_R. \end{cases} \tag{2.30}$$

It is clear from this definition that

$$v(t) = v_L(t) + v_R(t)\,. \tag{2.31}$$

We also write

$$\mathbf{T}_L(v) \equiv \int_{C_L} \frac{dt}{2\pi i}\, v(t)\, T(t) = \mathbf{T}(v_L)\,, \qquad \mathbf{T}_R(v) \equiv \int_{C_R} \frac{dt}{2\pi i}\, v(t)\, T(t) = \mathbf{T}(v_R)\,, \tag{2.32}$$

leading to the relation

$$\mathbf{T}_L(v) + \mathbf{T}_R(v) = \mathbf{T}(v)\,. \tag{2.33}$$

Given a vector $v(t)$, we define the *dual* vector field $\widetilde{v}(t)$ by reversing the sign of the left part of $v(t)$:

$$\widetilde{v}(t) \equiv v_R(t) - v_L(t)\,. \tag{2.34}$$

Analogously, the dual operator $\widetilde{\mathbf{T}}(v)$ is defined as

$$\widetilde{\mathbf{T}}(v) \equiv \mathbf{T}_R(v) - \mathbf{T}_L(v) = \mathbf{T}(\widetilde{v})\,. \tag{2.35}$$

Note that duality is an involution both for vectors and operators: applied twice it produces no change.

We have seen that BPZ even or odd vectors satisfy even or odd conditions under $t \to -t^*$. Since this reflection maps C_L and C_R into each other, the BPZ parity of a vector is changed when we change the sign of the vector over C_R or C_L. Therefore, if v has definite BPZ parity, its dual $\widetilde{v}$ will have opposite parity,

$$v = \pm v^\star \longrightarrow \widetilde{v} = \mp(\widetilde{v})^\star\,. \tag{2.36}$$

For explicit computations of dual of vector fields we can use the function $\epsilon(t)$ defined as

$$\epsilon(t) = \begin{cases} -1\,, & \text{if } t \in C_L; \\ +1\,, & \text{if } t \in C_R; \end{cases} \tag{2.37}$$

Multiplication of a vector $v(t)$ by $\epsilon(t)$ changes the sign of the left part of the vector and thus implements duality. The function ϵ has the Fourier series

$$\epsilon(e^{i\theta}) = \frac{2}{\pi} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{2k+1}\, e^{i(2k+1)\theta}\,, \tag{2.38}$$

or, equivalently, a Laurent series in t:

$$\epsilon(t) = \frac{2}{\pi} \sum_{k\in\mathbb{Z}} \frac{(-1)^k}{2k+1}\, t^{(2k+1)}\,. \tag{2.39}$$

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We note that

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^k}{2k+1} t^{2k+1} = \cdots + \frac{1}{5t^5} - \frac{1}{3t^3} + \frac{1}{t} + t - \frac{t^3}{3} + \frac{t^5}{5} + \cdots = \tan^{-1}(t) + \tan^{-1}\left(\frac{1}{t}\right) . \quad (2.40)$$

All in all

$$\widetilde{v}(t) = v(t)\epsilon(t) = v(t) \cdot \frac{2}{\pi}\left[\tan^{-1}(t) + \tan^{-1}\left(\frac{1}{t}\right)\right] . \quad (2.41)$$

When used to calculate a Virasoro operator, the factor in brackets must be Laurent expanded.

We now specialize this formalism to the operator $L^+ = L + L^\star$. Since $L + L^\star$ is BPZ even we can write

$$L^+ \equiv L + L^\star = \mathbf{T}(v^+) , \qquad \text{with} \quad (v^+)^\star(t) = v^+(t) . \quad (2.42)$$

The dual of L^+ is the important BPZ odd operator K,

$$K \equiv \widetilde{L^+} = L^+_R - L^+_L = \mathbf{T}(v^+_R) - \mathbf{T}(v^+_L) = \mathbf{T}(\widetilde{v^+}) . \quad (2.43)$$

Since duality is an involution, we also have

$$L^+ = \widetilde{K} = K_R - K_L , \quad \text{with } K_L = -L^+_L , \quad K_R = L^+_R . \quad (2.44)$$

2.3 When do L^+_L and L^+_R commute?

The function $\epsilon(t)$ must be manipulated with some care. We have, for example

$$\partial\epsilon = \frac{2}{\pi}\sum_{k\in\mathbb{Z}}(-1)^k t^{2k} = \frac{2}{\pi}\left(\ldots + \frac{1}{t^4} - \frac{1}{t^2} + 1 - t^2 + t^4 + \cdots\right) . \quad (2.45)$$

The right hand side is almost zero, as is appropriate for the almost constant function ϵ whose derivative is the sum of two delta functions:

$$\partial\epsilon = -2\delta\left(\theta - \frac{\pi}{2}\right) + 2\delta\left(\theta - \frac{3\pi}{2}\right) . \quad (2.46)$$

Note that $\partial\epsilon$ is a BPZ even vector and the associated stress-tensor is the BPZ even operator

$$\mathbf{T}(\partial\epsilon) = \frac{2}{\pi}\sum_{k=0}^{\infty}(-1)^{k+1}(L_{2k+1} - L_{-(2k+1)}) . \quad (2.47)$$

It is also useful to introduce the operator associated with t times $\partial\epsilon$:

$$\mathbf{T}(t\partial\epsilon) = \frac{2}{\pi}\left(L_0 + \sum_{k=1}^{\infty}(-1)^k(L_{2k} + L_{-2k})\right) . \quad (2.48)$$

One can see from the expansion (2.45) that $t^2 \cdot \partial\epsilon = -\partial\epsilon$. This is consistent with (2.46) since $t^2 = e^{2i\theta} = -1$ for $\theta = \pm\pi/2$. It follows that product of $\partial\epsilon$ and a function of t^2

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gives ϵ times the function evaluated at $t = i$.[6] The product of t times $\partial\epsilon$, however, is not proportional to $\partial\epsilon$, thus the necessity for the alternative operator (2.48). A function $\eta(t)$ with a Laurent expansion can be written as

$$\eta(t) = \eta_1(t) + t\eta_2(t)\,, \tag{2.49}$$

where both η_1 and η_2 contain only even powers of t. It then follows that

$$\eta(t)\partial\epsilon(t) = \eta_1(i)\,\cdot\partial\epsilon(t) + \eta_2(i)\cdot\, t\partial\epsilon(t)\,. \tag{2.50}$$

We say that $\eta(t)$ vanishes *strongly* at $t = i$ if both η_1 and η_2 vanish at $t = i$. If $\eta(t)$ vanishes strongly at $t = i$, then $\eta(t)\partial\epsilon = 0$. Finally, we note that $\epsilon(t)\cdot\epsilon(t) = 1$. This is manifest from the definition of ϵ, but can also be checked by explicit squaring of the power series (2.40).

We are now ready to discuss when property [**2a**] holds. First note that

$$[L^+, K] = [\,L_L^+ + L_R^+\,, L_R^+ - L_L^+] = 2\,[\,L_L^+\,, L_R^+\,]\,. \tag{2.51}$$

We thus ask, equivalently, when do L^+ and K commute? To answer this, we compute

$$[L^+, K] = [\,\mathbf{T}(v^+)\,, \mathbf{T}(\widetilde{v^+}\,)] = \mathbf{T}(\,[\,v^+, v^+\epsilon]\,)\,. \tag{2.52}$$

We now note that

$$[v^+\,, v^+\epsilon] = v^+\partial(v^+\epsilon) - v^+\epsilon\partial v^+ = (v^+)^2\partial\epsilon\,. \tag{2.53}$$

All in all

$$[L_L^+\,, L_R^+\,] = \frac{1}{2}\,\mathbf{T}((v^+)^2\partial\epsilon)\,. \tag{2.54}$$

The two operators commute if $(v^+)^2$ vanishes strongly at $t = i$. The vector fields v^+ we will consider are of the form $v^+ = tv_2(t)$, where v_2 is a function of t^2. In this case L_L^+ and L_R^+ commute when v^+ simply vanishes at $t = i$.

We do not know what additional conditions, if any, are needed for property [**2b**] to hold. The answer, of course, may depend on the class of states we use in (2.6). We leave this question unanswered.

We can readily consider more general commutators. For example, given two vectors v and w we examine

$$[\mathbf{T}_L(v)\,,\, \mathbf{T}_R(w)] = [\mathbf{T}(v_L)\,,\, \mathbf{T}(w_R)] = -\mathbf{T}\big(\,[v_L\,, w_R\,]\big)\,. \tag{2.55}$$

In order to compute the Lie bracket to the right, we note that

$$v_L = \frac{1}{2}(1-\epsilon)\,v\,, \qquad w_R = \frac{1}{2}(1+\epsilon)\,w\,, \tag{2.56}$$

and using $\epsilon^2 = 1$,

$$[v_L\,, w_R] = \frac{1}{2}\,vw\,\partial\epsilon\,. \tag{2.57}$$

[6]This property explains why (2.48) is BPZ even. For an arbitrary vector v of definite BPZ parity, tv does not have definite BPZ parity.

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As a result,

$$[\mathbf{T}_L(v)\,,\,\mathbf{T}_R(w)] = -\frac{1}{2}\mathbf{T}\big(vw\,\partial\epsilon\big)\,. \tag{2.58}$$

Note that (2.54) follows from (2.58) for when we set both v and w equal to v^+. Note now that for vectors v_1 and v_2 we have

$$[\widetilde{v}_1\,, v_2] = [v_1,\, v_2]^{\widetilde{\ }} - v_1 v_2\,\partial\epsilon\,, \tag{2.59}$$

where the tilde on the right-hand side acts on the full commutator. It then follows that the corresponding operators $\mathbf{T}(v_1)$ and $\mathbf{T}(v_2)$ satisfy

$$[\widetilde{\mathbf{T}}(v_1), \mathbf{T}(v_2)] = [\mathbf{T}(v_1), \mathbf{T}(v_2)]^{\widetilde{\ }} + \mathbf{T}(v_1 v_2\,\partial\epsilon)\,. \tag{2.60}$$

If we can ignore the midpoint contributions and the algebra [**1**] holds, we then have

$$[K, L] = [\widetilde{L^+}, L] = [L^+, L]^{\widetilde{\ }} = -\widetilde{L}^+ = -K\,. \tag{2.61}$$

Example: Given the BPZ odd derivation $K = \frac{\pi}{2}(L_1 + L_{-1})$ with vector $\frac{\pi}{2}(1+t^2)$, the corresponding BPZ even dual vector is

$$v^+(t) = (1+t^2)\cdot\left[\tan^{-1}(t) + \tan^{-1}\left(\frac{1}{t}\right)\right]. \tag{2.62}$$

We recognize here the vector corresponding to the sliver's L^+. Since $(v^+)^2 = \frac{\pi^2}{4}(1+t^2)^2$ vanishes for $t = i$, the operators L_L^+ and L_R^+ commute (see (2.54)). *Example:* For the butterfly

$$L^+ = \frac{1}{2}\,(\mathcal{L}_0 + \mathcal{L}_0^\star) = \frac{1}{2}L_{-2} + L_0 + \frac{1}{2}L_2 = \widetilde{K}\,, \tag{2.63}$$

for some suitable operator K. Here

$$v^+(t) = \frac{1}{2}\,t^3 + t + \frac{1}{2t}\,. \tag{2.64}$$

Clearly $v^+(t)$ is of the form $tv_2(t)$, with $v_2(t)$ an even function of t that vanishes for $t = i$. It follows that L_L^+ and L_R^+ commute. The vector dual to v^+ is

$$\widetilde{v^+}(t) = \left(\frac{1}{2}t^3 + t + \frac{1}{2t}\right)\frac{2}{\pi}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{2k+1}\,t^{2k+1}\,. \tag{2.65}$$

A short computation gives the simplified form

$$\widetilde{v^+}(t) = -\frac{8}{\pi}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{(4k^2-1)(2k+3)}\;t^{2k+2}\,. \tag{2.66}$$

It follows that

$$K = \mathbf{T}\,(\widetilde{v^+}) = -\frac{8}{\pi}\sum_{k\in\mathbb{Z}}\frac{(-1)^k}{(4k^2-1)(2k+3)}\;L_{2k+1} \;= -\frac{8}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(4k^2-1)(2k+3)}\;K_{2k+1}\,, \tag{2.67}$$

where $K_n \equiv L_n - (-1)^n L_{-n}$ are the familiar derivations of the $*$-algebra [39]. We can then verify explicitly that

$$[L^+\,, K\,] = \left[\,\frac{1}{2}L_{-2} + L_0 + \frac{1}{2}L_2\,,\; K\,\right] = 0\,. \tag{2.68}$$

This confirms that $[L_L^+\,,\,L_R^+\,] = 0$.

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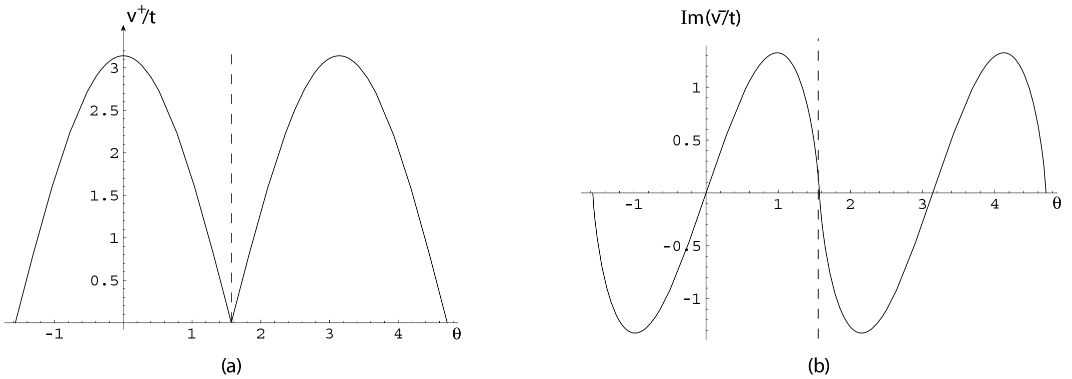

Figure 1: Plot of vectors associated with the sliver's L^+ and L^-. (a) Plot of $v^+(t)/t$, with $t = e^{i\theta}$, as a function of θ. (b) Plot of the imaginary part of $v^-(t)/t$ as a function of θ. Both v^+ and v^- vanish at the string midpoint $\theta = \pi/2$.

2.4 Derivations and the identity

We now discuss properties [**3a**] and [**3b**]. The derivations $K_n = L_n - (-1)^n L_{-n}$ are well known to kill the identity string field,

$$\langle \mathcal{I} | K_n = 0\,. \tag{2.69}$$

In the framework of conservation laws discussed in [7], for an operator $\mathbf{T}(v)$ to annihilate the identity the vector v must meet two conditions. First, it must be BPZ odd,

$$v(t) = \frac{v(-t^*)}{(t^*)^2}\,. \tag{2.70}$$

This condition states that v is consistent with the gluing condition of the identity. Second, when referred to the identity conformal frame $z = 2\xi/(1-\xi^2)$, the vector $v(z)$ must be an analytic function everywhere except at $z = 0$, where it may have poles. Both conditions can be checked for the vector $v_n(\xi) = \xi^{n+1} - (-1)^n \xi^{-n+1}$ corresponding to K_n.

Clearly, by the same argument, any finite linear combination of the K_n's will also kill the identity. Subtleties may arise when we consider infinite linear combinations. We believe that the analyticity condition for $v(z)$ that we just stated is stronger than needed: BPZ odd vectors with mild singularities still define operators that kill the identity. A proper understanding, which we will not attempt to provide, may require generalizing the framework of conservation laws to allow for certain kinds of non-analytic vector fields.

Let us illustrate this point for the case of the sliver. We start with the BPZ even vector v^+,

$$v^+(t) = (1+t^2)\left[\tan^{-1}(t) + \tan^{-1}\left(\frac{1}{t}\right)\right] = (1+t^2)\,\frac{\pi}{2}\epsilon(t)\,. \tag{2.71}$$

Because of the logarithmic branch cuts at $t = \pm i$, the domain of definition of v^+ is only the unit circle. Figure 1(a) shows a plot of $v^+(t)/t$ as a function of θ ($t = e^{i\theta}$). We note that there are corners at $\theta = \pm\pi/2$: the derivative fails to be continuous at this point. Interestingly, the singularities are erased in taking the dual vector $\widetilde{v^+} = \epsilon(t) v^+(t) = \frac{\pi}{2}(1+t^2)$,

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where we have used $\epsilon(t)^2 = 1$. The operator $K = \frac{\pi}{2} K_1$ certainly kills the identity – no issue here. On the other hand, the situation for L^- is more subtle. The corresponding BPZ odd vector v^- is

$$v^-(t) = (1+t^2) \left[\tan^{-1}(t) - \tan^{-1}\left(\frac{1}{t}\right) \right] . \tag{2.72}$$

This vector (plotted in figure 1(b)) vanishes at the string midpoint $t = \pm i$ where it has a first order zero multiplied by a divergent logarithm. The vector is only defined on the unit circle. Nevertheless, L^-, an infinite linear combinations of K_n derivations, actually kills the identity. We are going to prove this fact in section 3.2.

For the butterfly the situation is similar, with the roles of K and L^- in some sense reversed. The operators $L^\pm$ are finite linear combinations of Virasoro operators. The BPZ odd combination L^- is just K_2 and certainly kills the identity. While the singularity of the ϵ function shows up in the dual vector $\widetilde{v^+}$ that corresponds to K, it is quite mild: $\widetilde{v^+}/t$ and its derivative vanishes at the midpoint. Thus the butterfly K is even less singular than the sliver L^-, and we believe that it annihilates the identity.

We will encounter in section 7.3.1 examples of BPZ odd vector fields that, we suspect, fail to kill the identity. In those cases the midpoint singularity is a zero with fractional power. By contrast, in all concrete examples of special projectors that we are aware of, the BPZ odd vector fields have integer power zeroes and, at most, logarithmic singularities at the midpoint.

While condition [**2a**] led to a clear constraint on the midpoint behavior of the vector $v(\xi)$ associated with $\mathcal{L}_0$, we do not know what constraints on $v(\xi)$ are imposed by [**3a**] and [**3b**]. It may be that fractional power zeroes are not allowed for $v(\xi)$ anywhere on the unit circle.

3. Families of interpolating states

We now define a family of states parameterized by a real constant $\alpha \in [0, \infty)$:

$$\langle P_\alpha | \equiv \langle \mathcal{I} | \, e^{-\frac{\alpha}{2}(L+L^\star)} = \langle \mathcal{I} | \, e^{-\frac{\alpha}{2} L^+} . \tag{3.1}$$

Since the operator L^+ is BPZ even, we also have

$$|P_\alpha\rangle = e^{-\frac{\alpha}{2}(L+L^\star)} \, |\mathcal{I}\rangle = e^{-\frac{\alpha}{2} L^+} |\mathcal{I}\rangle . \tag{3.2}$$

Unlike the generic elements in the abelian subalgebra $\mathcal{A}_f$, the states $|P_\alpha\rangle$ have a geometric interpretation as surface states. If the operator L is defined using the conformal frame of the sliver the $|P_\alpha\rangle$ states are simply the familiar wedge states: $|n\rangle = |P_{n-1}\rangle$, with $|n = 2\rangle = |P_1\rangle$ equal to the SL$(2, R)$ vacuum and with $|P_\infty\rangle$ the sliver state. In general, the family of states $|P_\alpha\rangle$ interpolates between the identity, for $\alpha = 0$, and the limit state $|P_\infty\rangle$. In this section we discuss properties of the above families of states.

3.1 Star multiplication in the family

As explained in section 2, if conditions [**2**] hold we have the abelian algebra

$$(L^+)^n |\mathcal{I}\rangle * (L^+)^m |\mathcal{I}\rangle = (L^+)^{m+n} |\mathcal{I}\rangle , \tag{3.3}$$

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and we immediately find

$$|P_\alpha\rangle * |P_\beta\rangle = |P_{\alpha+\beta}\rangle\,. \tag{3.4}$$

Notice that for this we do not need to assume the algebra [**1**]. We can also prove (3.4) directly. Using [**2a**], [**2b**], and [**3b**] we have

$$|P_\alpha\rangle = e^{-\frac{\alpha}{2}(L_L^+ + L_R^+)}\,|\mathcal{I}\rangle = e^{-\frac{\alpha}{2}L_L^+}e^{-\frac{\alpha}{2}L_R^+}\,|\mathcal{I}\rangle \;= e^{-\alpha\,L_L^+}|\mathcal{I}\rangle = e^{-\alpha\,L_R^+}\,|\mathcal{I}\rangle\,, \tag{3.5}$$

and therefore

$$|P_\alpha\rangle * |P_\beta\rangle = (e^{-\alpha\,L_L^+}|\mathcal{I}\rangle) * |P_\beta\rangle = e^{-\alpha\,L_L^+}\,\big(|\mathcal{I}\rangle * |P_\beta\rangle\big) = e^{-\alpha\,L_L^+}|P_\beta\rangle = |P_{\alpha+\beta}\rangle\,, \tag{3.6}$$

as we wanted to show. From here we obtain a slightly generalized version of (3.3) as follows. First note that

$$L^+|P_\alpha\rangle = -2\frac{d}{d\alpha}\,|P_\alpha\rangle\,. \tag{3.7}$$

We then have

$$\begin{aligned}(L^+)^m|P_\alpha\rangle \,*\, (L^+)^n|P_\beta\rangle &= \left(-2\frac{d}{d\alpha}\right)^m |P_\alpha\rangle * \left(-2\frac{d}{d\beta}\right)^n |P_\beta\rangle \\ &= \left(-2\frac{d}{d\alpha}\right)^m \left(-2\frac{d}{d\beta}\right)^n [|P_\alpha\rangle * |P_\beta\rangle] \\ &= \left(-2\frac{d}{d\alpha}\right)^m \left(-2\frac{d}{d\beta}\right)^n |P_{\alpha+\beta}\rangle = (L^+)^{m+n}\,|P_{\alpha+\beta}\rangle\,.\end{aligned} \tag{3.8}$$

In summary,

$$(L^+)^m|P_\alpha\rangle \,*\, (L^+)^n|P_\beta\rangle = (L^+)^{m+n}\,|P_{\alpha+\beta}\rangle\,. \tag{3.9}$$

For $\alpha = \beta = 0$ we recover the expected (3.3).

3.2 Conservation laws

If we now assume the algebra [**1**] we can derive a useful conservation law for the states $\langle P_\alpha|$, namely, we find an interesting operator that annihilates the states.

Using [**3a**],

$$0 = \langle\mathcal{I}|L^- e^{-\frac{\alpha}{2}L^+} = \langle P_\alpha|e^{\frac{\alpha}{2}L^+}L^- e^{-\frac{\alpha}{2}L^+}\,. \tag{3.10}$$

The conjugation of L^- is readily evaluated since $[L^+, L^-] = -2L^+$ and $[L^+,[L^+,L^-]]$ vanishes:

$$0 = \langle P_\alpha|\,[\,L^- - \alpha L^+\,] = \langle P_\alpha|[\,(1-\alpha)L - (1+\alpha)L^\star\,]\,. \tag{3.11}$$

This is our conservation law. Taking BPZ conjugate, it is equivalently rewritten as

$$0 = \big[(1-\alpha)L^\star - (1+\alpha)\,L\big]\;|P_\alpha\rangle\,. \tag{3.12}$$

For $\alpha = 1$ this gives

$$L|P_1\rangle = 0\,. \tag{3.13}$$

For the sliver-based family, the state $|P_1\rangle$ is the $\mathrm{SL}(2,R)$ vacuum $|0\rangle$. Note that the $\mathrm{SL}(2,R)$ vacuum cannot belong to any other family of states whose limit state $|P_\infty\rangle$ is not the sliver. In fact given two families with different limit states, their only common state is the identity.

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Using (3.7) and the conservation law (3.12) one can easily verify that both L and $L^\star$ have simple action on $|P_\alpha\rangle$:

$$\begin{aligned} L|P_\alpha\rangle &= \ (\alpha-1)\,\frac{d}{d\alpha}\,|P_\alpha\rangle\,, \\ L^\star|P_\alpha\rangle &= -(\alpha+1)\frac{d}{d\alpha}\,|P_\alpha\rangle\,. \end{aligned} \tag{3.14}$$

In the above, we have assumed that L^- kills the identity and derived the conservation law (3.12). We could also run this logic backwards, taking the conservation law for some value of α as our axiom and *deduce* $L^-|\mathcal{I}\rangle = 0$. For example, taking $L|P_1\rangle = 0$ as the starting point, we have

$$0 = e^{\frac{1}{2}L^+}L\,|P_1\rangle = e^{\frac{1}{2}L^+}L\,e^{-\frac{1}{2}L^+}\,|\mathcal{I}\rangle = \frac{1}{2}L^-\,|\mathcal{I}\rangle\,. \tag{3.15}$$

In the case of the sliver, the statement that $L|P_1\rangle = L|0\rangle = 0$ is obvious. Thus for the sliver we have a simple proof that $L^-|\mathcal{I}\rangle = 0$.

3.3 The limit state $|P_\infty\rangle$

It is interesting to note that the family of states $|P_\alpha\rangle$ in (3.1) satisfies the abelian multiplication rule for *any* projector chosen to build the operators $\mathcal{L}_0$ and $\mathcal{L}_0^\star$: there was no need to assume [$\underline{\mathbf{1}}$] in order to prove (3.4). This raises an interesting question: What is the limit state $|P_\infty\rangle$? In this section we prove that the limit state is the projector itself when the algebra [$\underline{\mathbf{1}}$] holds. If [$\underline{\mathbf{1}}$] does not hold we do not know what $|P_\infty\rangle$ is, or if it is well defined. We have verified in section 6.1 that for a class of projectors that are not special $|P_\infty\rangle$ is not the projector used to define the family.

3.3.1 Surface states and $\mathcal{L}_0$

To begin our analysis we first discuss a property that shows the relevance of $\mathcal{L}_0$ to the construction of surface states. We want to prove that for *arbitrary* conformal frame $f(\xi)$ (not even a projector!), we can use the corresponding $\mathcal{L}_0$ to write the surface state $\langle f|$ as

$$\langle f| = \lim_{\gamma\to\infty}\langle\Sigma|\,e^{-\gamma\mathcal{L}_0}\,, \tag{3.16}$$

where $\langle\Sigma|$ is an arbitrary surface state. In order to establish this fact, we must recall how surface states are written in terms of exponentials of Virasoro operators.

Recall that given a conformal map $\xi\to f(\xi)$ the operator U_f that implements (via conjugation) the map takes the form $U_f = \exp(\mathbf{T}(\mathrm{v}))$ where the vector field $\mathrm{v}(\xi)$ is related to $f(\xi)$ by the Julia equation $\mathrm{v}(\xi)\partial_\xi f(\xi) = f(\mathrm{v}(\xi))$ [40]. In this case, the surface state $\langle f|$ associated with the function f is given by $\langle f| = \langle 0|U_f$. Given $\mathrm{v}(\xi)$ the function $f(\xi)$ is constructed as $f(\xi) = g^{-1}(1+g(\xi))$, where $g'(\xi) = 1/\mathrm{v}(\xi)$ [17]. We also recall the composition law $U_fU_g = U_{f\circ g}$ and note that the scaling $\xi\to f(\xi) = e^b\xi$ is realized by the operator e^{bL_0}.

We now establish a result that will help us prove (3.16) and will also have further utility.

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Claim: The operator $e^{-\gamma\mathcal{L}_0}$, with $\mathcal{L}_0$ defined by $f(\xi)$ in (2.1), realizes the conformal map

$$\xi \to h_\gamma(\xi) = f^{-1}(e^{-\gamma} f(\xi)) \,, \quad \text{or} \quad h_\gamma = f^{-1} \circ e^{-\gamma} \circ f \,. \tag{3.17}$$

Proof: First read the value of the vector v and use the algorithm described above to determine h_γ. Since $U_{h_\gamma} = \exp \mathbf{T}[v] = e^{-\gamma\mathcal{L}_0}$ we see that

$$\mathrm{v}(\xi) = -\gamma \frac{f(\xi)}{f'(\xi)} \,. \tag{3.18}$$

Then,

$$\frac{dg}{d\xi} = -\frac{1}{\gamma}\frac{f'(\xi)}{f(\xi)} \quad \to \quad g(\xi) = -\frac{1}{\gamma} \ln f(\xi) \,. \tag{3.19}$$

The inverse function to g is

$$g^{-1}(z) = f^{-1}(e^{-\gamma z}) \,, \tag{3.20}$$

so we get

$$h_\gamma(\xi) = g^{-1}(1 + g(\xi)) = f^{-1}\big(e^{-\gamma}\, e^{\ln f(\xi)}\big) = f^{-1}\big(e^{-\gamma} f(\xi)\big) \,. \tag{3.21}$$

This completes the proof of the claim. Note that (3.17) implies

$$U_{h_\gamma} = U_{f^{-1}} \cdot e^{-\gamma L_0} \cdot U_f \,. \tag{3.22}$$

Let us now return to (3.16). Writing $\langle\Sigma| = \langle 0|U_\Sigma$, we have

$$\begin{aligned} \lim_{\gamma\to\infty} \langle\Sigma|\, e^{-\gamma\mathcal{L}_0} &= \lim_{\gamma\to\infty} \langle 0|\, U_\Sigma \cdot U_{f^{-1}} \cdot e^{-\gamma L_0} \cdot U_f \\ &= \lim_{\gamma\to\infty} \langle 0|(e^{\gamma L_0} \cdot U_\Sigma \cdot e^{-\gamma L_0}) \cdot (e^{\gamma L_0} \cdot U_{f^{-1}} \cdot e^{-\gamma L_0}) \cdot U_f \end{aligned} \tag{3.23}$$

For any function $g(\xi) = a_g\, \xi + \mathcal{O}(\xi^2)$ one has

$$U_g = e^{(\ln a_g) L_0} \cdot \exp\left(\sum_{n=1}^{\infty} \gamma_n L_n\right) \,, \tag{3.24}$$

with some constants γ_n. We now note that

$$\lim_{\gamma\to\infty} e^{\gamma L_0} \cdot U_g \cdot e^{-\gamma L_0} = e^{(\ln a_g) L_0} \,. \tag{3.25}$$

Indeed, since

$$e^{\gamma\, L_0} \cdot L_n \cdot e^{-\gamma\, L_0} = e^{-n\,\gamma}\, L_n \,, \tag{3.26}$$

all positively moded Virasoro operators in the expansions of U_g are suppressed in the limit $\gamma \to \infty$. Back in (3.23) with $\Sigma(\xi) = a_\Sigma \xi + \ldots$ and $f^{-1}(\xi) = \frac{1}{a_f}\xi + \ldots$,

$$\lim_{\gamma\to\infty} \langle\Sigma|\, e^{-\gamma\mathcal{L}_0} = \langle 0|\, e^{(\ln a_\Sigma) L_0}\, e^{-(\ln a_f) L_0} \cdot U_f = \langle 0|\, U_f = \langle f| \,, \tag{3.27}$$

as we wanted to show. Given this representation, it is clear that the state $\langle f|$ is unchanged under the action of $e^{-\eta\mathcal{L}_0}$ for infinitesimal η. This implies that $\langle f|\mathcal{L}_0 = 0$. This is a familiar conservation law: the state $\langle f|$ represents the vacuum in the f conformal frame and $\mathcal{L}_0$ is the zero-mode Virasoro operator in that frame.

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3.3.2 Ordering the states $|P_\alpha\rangle$

To understand better the surface states $|P_\alpha\rangle$ we have to order the exponential $\exp(-\frac{\alpha}{2}L^+)$ in (3.1). The operator L^+ is double sided; it contains both positively and negatively moded Virasoro operators. Surface states, however, are usually written as exponentials of single sided operators acting on the vacuum. When presented as bras, the exponentials include Virasoro operators L_n with $n \geq 0$. Such a presentation is known for the identity, so we must now trade the operator $\exp(-\frac{\alpha}{2}L^+)$ for an operator that involves only L. This can be done if we continue to assume that the operators L and $L^\star$ satisfy the algebra [**1**].

We aim to decompose the Virasoro exponential as

$$e^{-\frac{\alpha}{2}(L+L^\star)} = x^{L^-}\, y^L\,, \tag{3.28}$$

where the constants x and y are determined by the value of α. Since $L^- = L - L^\star$ kills the identity, this will imply that

$$\langle P_\alpha| \equiv \langle \mathcal{I}|\, y^L\,. \tag{3.29}$$

We determine x and y as follows. Using (A.11) equation (3.28) is rewritten as

$$e^{-\frac{\alpha}{2}(L+L^\star)} = \left(\frac{2}{1+x^2}\right)^{L^\star} \left(\frac{2x^2 y}{1+x^2}\right)^{L}\,. \tag{3.30}$$

Comparing with the last equation in (A.12) we deduce that

$$x^2 = 1/y \quad \text{and} \quad \frac{2y}{1+y} = \frac{1}{1+\frac{\alpha}{2}} \quad \to \quad y = \frac{1}{1+\alpha}\,. \tag{3.31}$$

We therefore have

$$\langle P_\alpha| \equiv \langle \mathcal{I}|\, e^{-\gamma L}\,, \quad \text{with} \quad \gamma = \ln(1+\alpha)\,. \tag{3.32}$$

This provides a conventional presentation for the surface state $\langle P_\alpha|$.

3.3.3 Conformal frames and the state $|P_\infty\rangle$

We now wish to determine the conformal frame $z = f_\alpha(\xi)$ associated with the surface state $\langle P_\alpha|$. With this information we will be able to discuss the limit state $\langle P_\infty|$.

For the identity state we have

$$\langle \mathcal{I}| = \langle 0|U_{f_I}\,, \qquad f_I(\xi) = \frac{\xi}{(1-\xi^2)}, \tag{3.33}$$

so, for the $\langle P_\alpha|$ states (3.32) we have

$$\langle P_\alpha| = \langle 0|U_{f_I}U_{h_\gamma} = \langle 0|U_{f_I\circ h_\gamma}\,, \quad \text{with} \quad U_{h_\gamma} = e^{-\gamma L} = e^{-\frac{\gamma}{s}\mathcal{L}_0}\,. \tag{3.34}$$

It follows from the claim (3.17) proven earlier that

$$\langle P_\alpha| = \langle f_\alpha|\,, \quad \text{with} \quad f_\alpha = f_I \circ f^{-1} \circ e^{-\frac{\gamma}{s}} \circ f\,, \tag{3.35}$$

or for the corresponding operators,

$$\langle P_\alpha| = \langle 0|U_{f_\alpha}\,, \quad \text{with} \quad U_{f_\alpha} = U_{f_I}\cdot U_{f^{-1}}\cdot e^{-\frac{\gamma}{s}L_0}\cdot U_f\,. \tag{3.36}$$

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In order to examine the large α or, equivalently, the large γ limit we rewrite (3.36) as

$$U_{f_\alpha} = e^{-\frac{\gamma}{s}L_0} \cdot \Big(e^{\frac{\gamma}{s}L_0} \cdot U_{f_I} \cdot e^{-\frac{\gamma}{s}L_0}\Big) \cdot \Big(e^{\frac{\gamma}{s}L_0} \cdot U_{f^{-1}} \cdot e^{-\frac{\gamma}{s}L_0}\Big) \cdot U_f \,. \tag{3.37}$$

Since f_I goes like $f_I(\xi) = \xi + \mathcal{O}(\xi^2)$ and $f^{-1}(\xi) = \frac{1}{a_f}\xi + \mathcal{O}(\xi^2)$, we find

$$U_{f_\alpha} \simeq e^{-\left(\ln a_f + \frac{\gamma}{s}\right)L_0} \cdot U_f \,, \quad \text{as} \quad \alpha \to \infty \tag{3.38}$$

Using (3.36) we finally conclude that

$$\langle P_\alpha| = \langle 0|U_{f_\alpha} \simeq \langle 0|\, e^{-\left(\ln a_f + \frac{\gamma}{s}\right)L_0} \cdot U_f = \langle 0|U_f = \langle f| \quad \text{as} \quad \alpha \to \infty \,. \tag{3.39}$$

This is an important result: the limit state $\langle P_\infty|$ of the family is the surface state $\langle f|$ whose frame $f(\xi)$ was used to define $\mathcal{L}_0$. Let us recapitulate the logic. Starting with a conformal frame $f(\xi)$ obeying conditions [**2**] and [**3b**], we were able to construct the family of states $|P_\alpha\rangle$ obeying the abelian algebra (3.4). The algebra implies that if the limiting state $|P_\infty\rangle$ exists, it is a projector. By further assuming condition [**1**], we showed how to reorder the states $|P_\alpha\rangle$ into well-defined surface states, and further proved that $|P_\infty\rangle = |f\rangle$. All in all, we reach the nontrivial conclusion that if the map $f(\xi)$ satisfies [**1**], [**2**] and [**3b**], it defines a projector, indeed a special projector in our terminology. The role of condition [**3a**] is to guarantee that $\mathcal{L}_0$ has a simple action on $|P_\alpha\rangle$, which is necessary for solvability.

Finally, note that the limit state representation (3.39) implies that $\langle f|$ is invariant under the action of $e^{-\eta L^+}$, with infinitesimal η. Thus we see that $\langle f|L^+ = 0$. Since any (bra) surface state is annihilated by its $\mathcal{L}_0$, we conclude that special projectors are also annihilated by $\mathcal{L}_0^\star$.

4. Solving equations

We will consider here the string field equation

$$(\mathcal{L}_0 - 1)\Phi + \Phi * \Phi = 0 \,. \tag{4.1}$$

The operator $\mathcal{L}_0 = L_0 + \cdots$, together with $\mathcal{L}_0^\star$ will be assumed to satisfy the algebra [**1**]. As we will see, the solution is sensitive to the value of the constant s.

Before starting let us make a comment concerning reparameterizations. Under a midpoint preserving reparameterization (a symmetry of the theory) the string field transforms as

$$\Phi = e^{-K}\Phi' \,, \tag{4.2}$$

where the generator K is BPZ odd. Under this change (4.1) becomes

$$(\mathcal{L}_0' - 1)\Phi' + \Phi' * \Phi' = 0 \,, \qquad \text{with} \qquad \mathcal{L}_0' \equiv e^K \mathcal{L}_0\, e^{-K} \,. \tag{4.3}$$

It then follows that

$$\mathcal{L}_0'^{\,\star} = e^K \mathcal{L}_0^\star\, e^{-K} \,. \tag{4.4}$$

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Since they are related by similarity, the operators $\mathcal{L}_0, \mathcal{L}_0^\star$ and the operators $\mathcal{L}_0', \mathcal{L}_0'^\star$ define algebras with the *same* value of s. So solutions of the string field equation for different values of s are not related by reparameterization. Solutions for different $\mathcal{L}_0$ operators that have the same value of s might be related by reparameterizations, but we will not attempt to investigate this here.

In order to consider all values of s simultaneously, we use $L = \mathcal{L}_0/s$ and write (4.1) as

$$(s\,L-1)\Phi + \Phi * \Phi = 0\,. \tag{4.5}$$

All the s dependence is now in the kinetic term — recall that the L and $L^\star$ operators satisfy a universal, s independent algebra.

To compare our approach with that of Schnabl let us first recall how the $s=1$ equation was solved in [1]. An ansatz was given of the form

$$\Phi = \sum_{n=0}^{\infty} \frac{f_n}{n!}\left(-\frac{1}{2}\right)^n (L^+)^n|0\rangle\,, \qquad s=1\,. \tag{4.6}$$

where L^+ is understood to be the sliver's L^+, and the f_n are constants to be determined. For arbitrary special projectors, $|P_1\rangle$ plays the role that $|0\rangle$ plays for the sliver (see (3.13)). Thus, for arbitrary s the above must be replaced by

$$\Phi = \sum_{n=0}^{\infty} \frac{f_n}{n!}\left(-\frac{1}{2}\right)^n (L^+)^n|P_1\rangle\,. \tag{4.7}$$

Following the steps in [1] we would obtain the recursion

$$(s\,n-1)f_n = -\sum_{p+q\le n} \frac{n!}{p!\,q!\,(n-p-q)!}\, f_p\, f_q\,. \tag{4.8}$$

For $s=1$ the coefficients that emerge were recognized as Bernoulli numbers. For arbitrary s, the first few recursions above give

$$\begin{aligned} f_0 &= 1\,,\\ f_1 &= -\frac{1}{s+1}\,,\\ f_2 &= \frac{1+2s-s^2}{(1+s)^2(1+2s)}\,,\\ f_3 &= -\frac{(s-1)(2s^3-9s^2-6s-1)}{(1+s)^3(1+2s)(1+3s)}\,. \end{aligned} \tag{4.9}$$

For $s=1$ we recover the Bernoulli numbers $f_0=1, f_1=-\frac{1}{2}\,, f_2=\frac{1}{6}, f_3=0\,,\ldots$, while for $s=2$ we find the unfamiliar sequence $f_0=1, f_1=-\frac{1}{3}\,, f_2=\frac{1}{45}, f_3=\frac{11}{315}\,, f_4=\frac{29}{4725}$, etc. The $f_n(s)$ certainly define some deformation of the Bernoulli numbers, but it seemed difficult to obtain a full solution using this idea.

For $s=1$, equation (4.8) is a variant of the Euler relation for Bernoulli numbers, which expresses higher Bernoulli numbers in terms of products of lower ones. The Euler relation can be quickly derived using a differential equation satisfied by the generating function of the Bernoulli numbers.[7] To solve (4.5) we will derive differential equations for functions of

[7]We wish to thank J. Goldstone for explaining this to us.

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L^+. For $s = 1$ the solution will give directly the generating function of Bernoulli numbers. For other values of s the solution will be written in terms of confluent hypergeometric functions.

4.1 Deriving the differential equation

We will write the solution to (4.5) as an arbitrary function f_s of L^+ acting on the identity string field:

$$\Phi = f_s(x)|\mathcal{I}\rangle\,, \quad x \equiv L^+\,. \tag{4.10}$$

Let us first examine the kinetic term. We re-write (2.13) as

$$L\,x^n\,|\mathcal{I}\rangle = \left(n\,x^n + \frac{1}{2}x^{n+1}\right)\,|\mathcal{I}\rangle, \tag{4.11}$$

so acting on functions of x:

$$Lf(x)|\mathcal{I}\rangle = \left(x\frac{df}{dx} + \frac{x}{2}\,f\right)\,|\mathcal{I}\rangle\,. \tag{4.12}$$

Let us now examine the quadratic term. We have (see (2.16))

$$x^m|\mathcal{I}\rangle\,*\,x^n|\mathcal{I}\rangle = x^{m+n}\,|\mathcal{I}\rangle\,, \tag{4.13}$$

so for functions we find

$$f(x)|\mathcal{I}\rangle\,*\,g(x)|\mathcal{I}\rangle = f(x)g(x)\,|\mathcal{I}\rangle\,. \tag{4.14}$$

With these results, equation (4.5) becomes the following differential equation for $f_s(x)$:[8]

$$\left[s\,x\left(\frac{d}{dx} + \frac{1}{2}\right) - 1\right]\,f_s(x) + f_s^2(x)\,=0\,. \tag{4.16}$$

We are interested mostly in solutions of (4.1) for which

$$\Phi = |0\rangle + \text{Virasoro descendants}\,. \tag{4.17}$$

Indeed, if Φ has any component along the $\mathrm{SL}(2,R)$ vacuum state $|0\rangle$, the coefficient of this component must be equal to one. This happens because $|0\rangle * |0\rangle = |0\rangle +$ descendants, and, neither the star product of descendants nor the star product of a descendant and the vacuum contain the vacuum.

We now claim that $x^n|\mathcal{I}\rangle$, with n positive does not contain the vacuum $|0\rangle$. First note that, with zero central charge, the action of L^+ on a descendant gives a descendant. Second, $L^+|0\rangle$ is a descendant. Since the identity state is the vacuum plus a descendant, it follows that $L^+|\mathcal{I}\rangle$ is a descendant, and so is $(L^+)^n|\mathcal{I}\rangle$ for any integer $n \geq 1$. We assume that $f_s(x)$ has a Taylor expansion around $x = 0$, so we can conclude that the coefficient of $|0\rangle$ in the solution $f_s(x)|\mathcal{I}\rangle$ is $f_s(x = 0)$. Therefore, (4.17) holds if

$$f_s(x = 0) = 1\,. \tag{4.18}$$

[8] Had we set $\Phi = h_s(x)|P_1\rangle$, where $|P_1\rangle = e^{-x/2}|\mathcal{I}\rangle$, the differential equation would be

$$\left(s\,x\frac{d}{dx} - 1\right)h_s(x) + h_s^2(x)\,e^{-\frac{x}{2}} = 0\,. \tag{4.15}$$

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4.2 Solving the differential equation

Let us first consider the case $s = 1$, which applies to the sliver and must reproduce the result of [1]. Letting $f_1 = x/a(x)$ we get:

$$\frac{da}{dx} = \frac{1}{2}\, a + 1 \quad \rightarrow \quad a(x) = C\, e^{\frac{x}{2}} - 2\,, \tag{4.19}$$

where C is an integration constant. The condition (4.18) requires $C = 2$, which gives

$$\Phi = f_1(x)|\mathcal{I}\rangle = \frac{x/2}{e^{x/2} - 1}\, |\mathcal{I}\rangle\,. \tag{4.20}$$

To compare with [1], we use $|\mathcal{I}\rangle = e^{x/2}|P_1\rangle$ to write

$$\Phi = \frac{(-x/2)}{e^{-x/2} - 1}\, |P_1\rangle\,. \tag{4.21}$$

The function in front of $|P_1\rangle$ is then recognized as the generating function for Bernoulli numbers:

$$\frac{(-x/2)}{e^{-x/2} - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!}\left(-\frac{x}{2}\right)^n \quad \rightarrow \quad \Phi = \sum_{n=0}^{\infty} \frac{B_n}{n!}\left(-\frac{x}{2}\right)^n |P_1\rangle\,. \tag{4.22}$$

This is exactly the result in (4.7), with $f_n = B_n$, as it was found in [1]. Note the curious fact that the form (4.20) of the solution based on the identity also has an expansion governed by Bernoulli numbers, one that does not have the additional minus signs of (4.22):

$$\Phi = f_1(x)|\mathcal{I}\rangle = \sum_{n=0}^{\infty} \frac{B_n}{n!}\left(\frac{x}{2}\right)^n |\mathcal{I}\rangle\,. \tag{4.23}$$

Choosing $C = 2/\lambda$ one finds

$$f_1(x) = \frac{\lambda\,(x/2)}{e^{x/2} - \lambda}\,, \tag{4.24}$$

which, for $\lambda < 1$, corresponds to the "pure-gauge" solutions of [1] and do not contain a component along the vacuum state $|0\rangle$. For $\lambda = 1$, the solution (4.24) coincides with (4.20).

For arbitrary values of s we can solve (4.16) by writing

$$f_s(x) = \frac{x^{1/s}}{a_s(x)}\,. \tag{4.25}$$

We then obtain the first order ordinary differential equation

$$a_s' - \frac{1}{2}\, a_s = \frac{1}{s}\, x^{\frac{1}{s}-1} \quad \rightarrow \quad a_s(x) = a_s(0) + \frac{1}{s}\, e^{\frac{x}{2}} \int_0^x u^{\frac{1}{s}-1} e^{-\frac{u}{2}}\, du\,. \tag{4.26}$$

Letting $u = xt$ we obtain

$$a_s(x) = a_s(0) + e^{\frac{x}{2}} x^{\frac{1}{s}} \int_0^1 dt\, e^{-\frac{xt}{2}} \frac{d}{dt} t^{\frac{1}{s}}\,. \tag{4.27}$$

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To ensure (4.17) we demand $f_s(x) \to 1$ as $x \to 0$. This requires $a_s(x) \to x^{1/s}$ which, in turn, requires $a_s(0) = 0$. For $a_s(0) \neq 0$ the solution's leading behavior is $f_s(x) \sim x^{1/s}$ — this is supposed to be the "pure-gauge" solution. So, nontrivial solutions are

$$f_s(x) = \left[e^{\frac{x}{2}} \int_0^1 dt\, e^{-\frac{xt}{2}} \frac{d}{dt} t^{\frac{1}{s}} \right]^{-1} . \tag{4.28}$$

The integral (which is an incomplete Gamma function) can be readily transformed into a series by successive integration by parts. One finds

$$f_s(x) = \left[{}_1F_1\left(1, 1 + \frac{1}{s}, \frac{x}{2}\right)\right]^{-1} , \tag{4.29}$$

where ${}_1F_1$ is the confluent hypergeometric function with series expansion

$${}_1F_1(a, b, z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \cdots \tag{4.30}$$

One can write the solution of the string field equation as

$$\Phi_s = \left[{}_1F_1\left(1, 1 + \frac{1}{s}, \frac{x}{2}\right)\right]^{-1} |\mathcal{I}\rangle . \tag{4.31}$$

This is the solution of (4.1) for arbitrary $s > 0$.

Since ${}_1F_1(1, 2, \frac{x}{2}) = (e^{x/2} - 1)/(x/2)$, we recover the answer for $s = 1$. For other values of s the answer cannot be written in terms of elementary functions. For $s = 2$ we get

$$\begin{aligned} f_2(x) = \frac{2}{\sqrt{\pi}} \frac{e^{-x/2}\sqrt{x/2}}{\text{Erf}[\sqrt{x/2}]} &= 1 - \frac{x}{3} + \frac{2x^2}{45} - \frac{2x^3}{945} - \frac{2x^4}{14175} + \frac{2x^5}{93555} + \cdots \\ &= \left(1 + \frac{x}{6} + \frac{x^2}{360} - \frac{11x^3}{15120} + \frac{29x^4}{1814400} + \cdots \right) e^{-x/2} . \end{aligned} \tag{4.32}$$

The last form was included since it allows one to read the $s = 2$ coefficients $f_n(s)$ discussed below (4.9). For arbitrary s a series expansion of the solution gives

$$f_s(x) = 1 - \frac{s\,x}{2(1+s)} + \frac{s^3 x^2}{4(1+s)^2(1+2s)} - \cdots . \tag{4.33}$$

The limit $s \to \infty$ can be evaluated since the hypergeometric function becomes ${}_1F_1(1, 1, x/2) = \exp(x/2)$. The solution becomes

$$f_\infty(x) = e^{-x/2} , \qquad \Phi = e^{-x/2}|\mathcal{I}\rangle = |P_1\rangle , \quad s \to \infty . \tag{4.34}$$

Since $L|P_1\rangle = 0$ exactly, the equation is satisfied because $|P_1\rangle * |P_1\rangle = |P_2\rangle \sim |P_1\rangle$ in the $s \to \infty$ limit. In section 6.2 we will encounter an infinite family of special projectors, with $\mathcal{L}_0 = L_0 + (-1)^{m+1} L_{2m}$, $s = 2m$. It may appear that as $s \to \infty$, this sequence of operators converges the "Siegel gauge" operator L_0. This is probably naive — the operators L_0 and $L_0^\star$ coincide, and thus commute with each other, whereas in the proposed sequence the commutator becomes larger and larger. The solution of the "Siegel gauge" ghost number zero equation was computed in level truncation [31], and it is clearly not a surface state — unlike our large s solution $\Phi_\infty = |P_1\rangle$.

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4.3 Solution as a superposition of surface states

The solution based on the sliver can also be written as an infinite sum of derivatives of wedge states $|P_n\rangle$ plus an extra term sliver state [1]. Apart from that subtle extra term, the solution arises by naive expansion of the denominator of (4.21) in powers of $e^{-x/2}$:

$$\Phi = \frac{1}{2}\frac{x}{1-e^{-x/2}}\,|P_1\rangle = \frac{1}{2}\sum_{n=0}^{\infty} x\, e^{-nx/2}|P_1\rangle\,. \tag{4.35}$$

This expansion is not legal for $x = 0$, where $e^{-x/2} = 1$. Indeed for $x = 0$ the right-hand side of (4.35) vanishes term by term, while the left hand side is non-zero. Alternatively, the first expression for Φ contains the vacuum state while the second does not. The second expression can be used if we add to it the state $|P_\infty\rangle$:

$$\Phi = |P_\infty\rangle + \frac{1}{2}\sum_{n=1}^{\infty} L^+|P_n\rangle\,. \tag{4.36}$$

This is a solution that contains the vacuum state with unit coefficient. It can be viewed as the sum of two solutions. The first term $|P_\infty\rangle$ is a solution because it is annihilated by L and it is a projector. The second term is a solution because it inherits this property from the function that gave rise to it. The sum of solutions is a solution because $|P_\infty\rangle * L^+|P_\alpha\rangle = L^+|P_{\infty+\alpha}\rangle = L^+|P_\infty\rangle = 0$. If desired, L^+ can be viewed as a derivative of the surface state using (3.7). Interestingly, only wedge states $|P_n\rangle$ with $n \geq 1$ contribute. This means that, in some sense, the solution receives no contribution from the identity string field, nor from any wedge state $|P_\alpha\rangle$ with $\alpha < 1$.

We now explain how to write the arbitrary s solution as a superposition of $|P_\alpha\rangle$ states. We will also show that only states with $\alpha \geq 1$ contribute. Inspired by the structure of (4.36) we write

$$\Phi_s = |P_\infty\rangle + \int_0^\infty d\alpha\, \mu_s(\alpha)\, L^+|P_\alpha\rangle\,. \tag{4.37}$$

As in the case of $s = 1$, we identify the second term above with the solution obtained from the differential equation,

$$f_s(x)|\mathcal{I}\rangle = \int_0^\infty d\alpha\, \mu_s(\alpha)\, x\, e^{-\frac{\alpha}{2}x}|\mathcal{I}\rangle\,. \tag{4.38}$$

This equation requires

$$\frac{f_s(2x)}{2x} = \int_0^\infty d\alpha\, \mu_s(\alpha)\, e^{-\alpha x}\,, \tag{4.39}$$

which states that $f_s(2x)/(2x)$ is the Laplace transform of the density $\mu_s(\alpha)$.[9] Equation (4.39) implies a familiar property of Laplace transforms: if the right-hand side integral converges for some x_0 it converges for all x with $\mathrm{Re}\, x > \mathrm{Re}\, x_0$. This implies that the left-hand side must have an abscissa of convergence. In fact, we believe that $f_s(2x)/(2x)$ is finite for $\mathrm{Re}\, x > 0$.[10]

[9] In the usual notation for Laplace transforms $\mu(\alpha)$ corresponds to the signal $G(t)$, which vanishes for $t < 0$, and $f(2x)/(2x)$ corresponds to the Laplace transform $G(s)$.

[10] This happens if ${}_1F_1\big(1, 1 + \frac{1}{s}, x\big)$ has no zero for $\mathrm{Re}\, x > 0$. This is readily checked for $s = 1$.

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It follows from (4.39) that the density $\mu_s(\alpha)$ is the inverse Laplace transform of $f_s(2x)/(2x)$:

$$\mu_s(\alpha) = \int_{c-i\infty}^{c+i\infty} dx\, e^{\alpha x}\, \frac{f_s(2x)}{2x}\,. \tag{4.40}$$

The real constant c can be chosen to be any number greater than zero.

For large x with $\mathrm{Re}\, x > 0$ one has the asymptotic behavior:

$$\frac{f_s(2x)}{2x} \simeq \frac{1}{2\Gamma(1+1/s)}\,\frac{e^{-x}}{x^{1-\frac{1}{s}}}\,. \tag{4.41}$$

This relation, a textbook property of the confluent hypergeometric function, can be gleaned from eqs. (4.26) and (4.25), recalling that $a_s(0) = 0$. Since $f_s(2x)/(2x)$ is analytic for $\mathrm{Re}\, x > 0$, the contour integral in (4.40) can be deformed into a very large semicircle over which (4.41) applies. It follows that for $\alpha < 1$ and $s > 1$ the integral over the half-circle goes to zero as the radius of the circle goes to infinity, so we conclude that

$$\mu_s(\alpha) = 0 \quad \text{for} \quad \alpha < 1\,. \tag{4.42}$$

This is what we wanted to establish. We can perform the inverse Laplace transform of (4.41) getting

$$\mu_s(\alpha) \simeq \frac{1}{2}\frac{\sin\frac{\pi}{s}}{\frac{\pi}{s}}\,\frac{1}{(\alpha-1)^{1/s}}\,\Theta(\alpha-1)\,, \tag{4.43}$$

where $\Theta(u)$ is the step function: $\Theta(u) = 1$ for $u \geq 0$, $\Theta(u) = 0$ for $u < 0$. Note that the density vanishes for $\alpha < 1$ and it has an integrable singularity at $\alpha = 1$.

The case $s = 1$ is a bit special and the corresponding $\mu_1(\alpha)$ can be readily found:

$$\mu_1(\alpha) = \int_{c-i\infty}^{c+i\infty} dx\, e^{\alpha x}\, \frac{f_1(2x)}{2x} = \frac{1}{2}\int_{c-i\infty}^{c+i\infty} dx\, e^{\alpha x} \sum_{n=1}^{\infty} e^{-nx} = \frac{1}{2}\sum_{n=1}^{\infty}\delta(\alpha-n)\,, \tag{4.44}$$

which back in (4.37) reproduces (4.36).

The density $\mu_2(\alpha)$ for $s = 2$ can be calculated using an expansion around $x = \infty$. Using the asymptotic expansion

$$\mathrm{Erf}[\sqrt{x}] = 1 - \frac{e^{-x}}{\sqrt{\pi x}}\left(1 + \sum_{n=1}^{\infty}\frac{c_n}{x^n}\right)\,, \qquad c_n = \left(-\tfrac{1}{2}\right)^n (2n-1)!!\,, \tag{4.45}$$

as well as (4.32), we can write

$$\frac{f_2(2x)}{2x} = \frac{e^{-x}}{\sqrt{\pi x}}\,\frac{1}{\mathrm{Erf}[\sqrt{x}]} = \sum_{m=1}^{\infty}\left(\frac{e^{-x}}{\sqrt{\pi x}}\right)^m \left(1 + \sum_{n=1}^{\infty}\frac{c_n}{x^n}\right)^{m-1}\,. \tag{4.46}$$

The inverse Laplace transform can be organized by the exponentials $e^{-(m+1)x}$ each of which produces a $\Theta(\alpha-(m+1))$. It follows that $\mu_2(\alpha)$ can be calculated for $\alpha < 4$, for example, by using the terms with $m = 1, 2$, and 3. We get

$$\begin{aligned} \mu_2(\alpha) = {} & \frac{1}{\pi}\frac{\Theta(\alpha-1)}{\sqrt{\alpha-1}} + \frac{1}{\pi}\Theta(\alpha-2)\left(1 + \sum_{n=1}^{\infty}\frac{c_n}{n!}(\alpha-2)^n\right) \\ & + \frac{2}{\pi^2}\,\Theta(\alpha-3)\sqrt{\alpha-3}\left(1 + \sum_{n=1}^{\infty}\frac{2^n\, d_n}{(2n+1)!!}(\alpha-3)^n\right) + \mathcal{O}(\Theta(\alpha-4))\,, \end{aligned} \tag{4.47}$$

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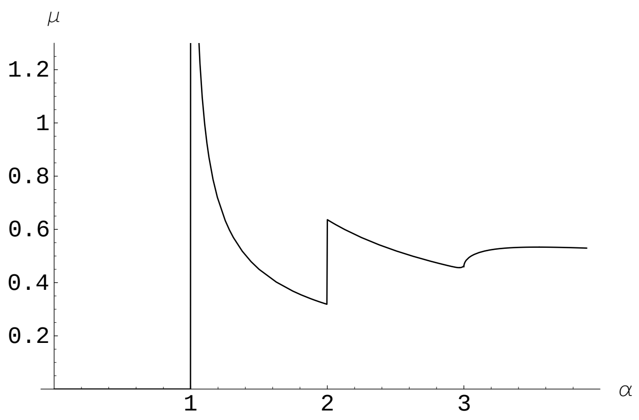

Figure 2: The density $\mu_2(\alpha)$ for the string field solution of (4.1) with $s = 2$. The density vanishes for $\alpha < 1$. It has an integrable singularity for $\alpha = 1$, it is discontinuous at $\alpha = 2$ and has a discontinuous derivative at $\alpha = 3$.

where the coefficients d_n are defined by

$$d_n = 2c_n + \sum_{k=1}^{n-1} c_k c_{n-k} \,. \tag{4.48}$$

A plot of the function $\mu_2(\alpha)$ is shown in figure 2.[11] We believe that $\mu_s(\alpha)$ for $s > 2$ behaves similarly: it can be built as a sum of layered step functions, the first of which is multiplied by a function with an integrable singularity at $\alpha = 1$.

4.4 Ordering algorithm and descendant expansion

In this subsection we show how to use the previously found solutions to obtain the exact coefficients of a descendant expansion of the string field. Since L^+ contains both positively moded and negatively moded Virasoro operators, the solution in the form

$$\Phi = f(x)|\mathcal{I}\rangle\,, \quad x = L^+\,, \tag{4.49}$$

is not suitable for direct evaluation in level expansion. Indeed, although a convenient level expansion for $|\mathcal{I}\rangle$ is available [41]

$$\begin{aligned} |\mathcal{I}\rangle &= \cdots \exp\Bigl(-\frac{1}{8}L_{-16}\Bigr) \exp\Bigl(-\frac{1}{4}L_{-8}\Bigr) \exp\Bigl(-\frac{1}{2}L_{-4}\Bigr) \exp(L_{-2})|0\rangle\,. \\ &= |0\rangle + L_{-2}|0\rangle + \frac{1}{2}L_{-2}L_{-2}|0\rangle - \frac{1}{2}L_{-4}|0\rangle + \cdots\,, \end{aligned} \tag{4.50}$$

the action of functions of L^+ on $|\mathcal{I}\rangle$ is complicated. We need to reorder the expansion by rewriting Φ as a function of $L^\star$, which does not contain positively moded Virasoro operators (it would simplify matters even further if we could remove the L_0 from $L^\star$). Thus we want to use the function f to calculate a function g such that

$$\Phi = f(x)|\mathcal{I}\rangle = g(u)|\mathcal{I}\rangle \quad \text{with} \quad u \equiv L^\star\,. \tag{4.51}$$

[11] Since the series in α that multiplies $\Theta(\alpha - 2)$ does not converge beyond $\alpha = 3$, we used (numerical) analytic continuation to construct it in the range $\alpha \in [3, 4]$.

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We know that $L|\mathcal{I}\rangle = L^\star|\mathcal{I}\rangle$. Therefore, acting on the identity,

$$x = L^+ = L + L^\star = 2L^\star = 2u\,. \tag{4.52}$$

Since $xu = (u+1)x$ we also have that, acting on the identity,

$$x^2 = x(2u) = 2(u+1)x = 2^2(u+1)u\,. \tag{4.53}$$

It readily follows from repeated application that, acting on the identity,

$$x^n = 2^n(u+n-1)(u+n-2)\ldots(u+1)u = 2^n\,P_n(u)\,. \tag{4.54}$$

We therefore have the ordering relation (on the identity)

$$x^n = 2^n\,\frac{\Gamma(u+n)}{\Gamma(u)}\,. \tag{4.55}$$

Note that, in fact, this relation holds for $n \geq 0$. Assuming the Taylor-expansion

$$f(x) = \sum_{n=0}^{\infty} f_n x^n\,, \tag{4.56}$$

and applying the reordering formula (4.55) to each term of the series, we find

$$g(u) = \frac{1}{\Gamma(u)}\sum_{n=0}^{\infty} f_n\,2^n\,\Gamma(u+n) = \frac{1}{\Gamma(u)}\int_0^\infty dt\,e^{-t}t^{u-1}\sum_{n=0}^{\infty} f_n\,(2t)^n\,. \tag{4.57}$$

We can thus write $g(u)$ as an integral transform of $f(x)$

$$g(u) = \frac{1}{\Gamma(u)}\int_0^\infty dt\;e^{-t}\,t^{u-1}f(2t)\,. \tag{4.58}$$

The above is actually the Mellin transform of the function $e^{-t}f(2t)$. One also verifies that

$$g(0) = f(0)\,, \tag{4.59}$$

as expected from the reordering algorithm.

Applying the reordering functional to our solutions (4.31) we find

$$\Phi_s = g_s(u)|\mathcal{I}\rangle = \frac{1}{\Gamma(u)}\int_0^\infty dt\;\frac{e^{-t}\,t^{u-1}}{{}_1F_1\left(1, 1+\frac{1}{s}, t\right)}\;\;|\mathcal{I}\rangle\,,\qquad u = L^\star\,. \tag{4.60}$$

This is our final result for the reordered solution of the string field equation of motion. We now consider special values of s to understand how the result can be used. For $s = 1$ the integral becomes

$$\Phi_1 = \frac{1}{\Gamma(u)}\int_0^\infty dt\;\frac{e^{-t}\,t^{u}}{e^t-1}\;\;|\mathcal{I}\rangle\,. \tag{4.61}$$

This is a familiar integral that can be evaluated in terms of ζ functions:

$$\Phi_1 = u\big(\zeta(u+1)-1\big)\;|\mathcal{I}\rangle\,,\qquad u = L^\star\,. \tag{4.62}$$

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This answer is consistent with the form found by Schnabl, whose solution is written as a function of $L^\star$ acting on $|P_1\rangle$. For other finite values of s one can calculate the function $g_s(u)$ numerically. For $s = 2$, for example, we obtain

$$g_2(1) = 0.584273\,, \qquad g_2(2) = 0.334025\,. \tag{4.63}$$

As $s \to \infty$ we find $g_\infty(u) = 2^{-u}$ and thus

$$\Phi_\infty = 2^{-u}\,|\mathcal{I}\rangle\,, \qquad u = L^\star\,. \tag{4.64}$$

The final stage of our calculation requires writing the string field $g(L^\star)|\mathcal{I}\rangle$ as an expansion in Virasoro descendants – the familiar level expansion. We denote

$$\Phi = g_s(L^\star)|\mathcal{I}\rangle = \gamma_0(s)|0\rangle + \gamma_2(s)\,L_{-2}|0\rangle + \gamma_4(s)\,L_{-4}|0\rangle + \gamma_{2,2}(s)\,L_{-2}L_{-2}|0\rangle + \cdots \tag{4.65}$$

with computable γ coefficients. In here not only is the value of s important but one also requires the explicit form of the operator $L^\star$. We write

$$sL^\star = L_0 + \alpha_2 L_{-2} + \alpha_4 L_{-4} + \cdots\,, \tag{4.66}$$

with constants $\alpha_2, \alpha_4, \ldots$ that take different values for the different conformal frames. We now obtain closed-form expressions for the first few γ coefficients in the descendant expansion. As a first step in the calculation one shows that for $n \geq 1$

$$\begin{aligned}(sL^\star)^n|\mathcal{I}\rangle &= 2^n\,\frac{1}{2}(\alpha_2+2)L_{-2}|0\rangle + 4^n\,\frac{1}{4}(\alpha_4-2)L_{-4}|0\rangle \\ &\quad + \left(4^n\,\frac{1}{8}(\alpha_2+2)^2 - 2^n\frac{1}{4}\alpha_2(\alpha_2+2)\right)\,L_{-2}L_{-2}|0\rangle + \cdots\end{aligned} \tag{4.67}$$

One way to obtain the above result is to work out explicitly $(sL^\star)^n|\mathcal{I}\rangle$ for $n = 1, 2, 3$, and 4 — the pattern then becomes clear. Assuming a Taylor expansion $g(u) = \sum_{n=0}^{\infty} g_n u^n$ around $u = 0$, the above result leads to (4.65) with

$$\begin{aligned}\gamma_0(s) &= \ g_s(0)\,, \\ \gamma_2(s) &= -\frac{1}{2}\left[\alpha_2\,g_s(0) - (\alpha_2+2)\,g_s\left(\frac{2}{s}\right)\right]\,, \\ \gamma_4(s) &= -\frac{1}{4}\left[\alpha_4\,g_s(0) - (\alpha_4-2)\,g_s\left(\frac{4}{s}\right)\right]\,, \\ \gamma_{2,2}(s) &= \ \frac{1}{8}\left[\alpha_2^2\,g_s(0) - 2\,\alpha_2(\alpha_2+2)\,g_s\left(\frac{2}{s}\right) + (\alpha_2+2)^2\,g_s\left(\frac{4}{s}\right)\right]\,.\end{aligned} \tag{4.68}$$

Since the function $g_s(u)$ can be calculated numerically for any s and u, the above descendant expansion can also be obtained numerically. For illustration we consider the case $s = 2$ with the butterfly operator $sL = L_0 + L_2$. We thus have $\alpha_2 = 1$ and $\alpha_4 = 0$. Together with $g_2(0) = 1$ and the values recorded in (4.63) we get

$$\begin{aligned}&s = 2,\ \mathcal{L}_0 = L_0 + L_2\ : \\ &\Phi = |0\rangle + 0.37641\,L_{-2}|0\rangle - 0.167012\,L_{-4}|0\rangle + 0.062573\,L_{-2}L_{-2}|0\rangle + \cdots\,.\end{aligned} \tag{4.69}$$

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5. $\ell^\star$-level expansion

Ordinary level truncation is not a very economical way to solve the string field equations (4.1). Indeed, the star multiplication of two Virasoro descendants of some fixed levels generally gives Virasoro descendants of *all* levels, multiplied by coefficients that take some effort to obtain. Moreover, convergence to the solutions seems rather slow. The exact solution obtained in section 4 was based on an expansion in powers of L^+ acting on the identity, the power is then called the level ℓ^+. The great advantage of this expansion is that the star product is exactly additive: $\ell^+(\Phi_1 * \Phi_2) = \ell^+(\Phi_1) + \ell^+(\Phi_2)$. This allowed us to solve the equation analytically. The remaining complication was the need to order the solution when a descendant expansion is needed. The way to order the solution was explained in section 4.4.

In this section we examine an alternative level truncation. We expand the string field in powers of $L^\star$ acting on the identity. The level $\ell^\star$ of a given term is defined to be the power of $L^\star$. This expansion, as we will see, has two advantages over ordinary L_0-level expansion in Virasoro descendants. First, the multiplication is *sub-additive*: $\ell^\star(\Phi_1 * \Phi_2) \leq \ell^\star(\Phi_1) + \ell^\star(\Phi_2)$. Second, the coefficients appearing in the product are simple to evaluate. Moreover, the kinetic term gives no trouble: the L action on a term of level l gives terms with level less than or equal to $l+1$. Of course, just like for ordinary level truncation, the recursions are not exactly solvable.

In the $L^\star$ expansion the string field is written as

$$\Phi = g_s(u)\,|\mathcal{I}\rangle = \Big(1 + a_1\,u + a_2 u^2 + a_3 u^3 + \cdots\Big)\,|\mathcal{I}\rangle\,. \tag{5.1}$$

As before, we use the variables $x = L^+$ and $u = L^\star$ as well as the useful relations $x\,u = (u+1)\,x$ and $u\,x = x(u-1)$ which allow us to move x's and u's across one another. On the identity $u = x/2$. In order to evaluate the kinetic term we note that

$$L\,u^n = (x-u)u^n = xu^n - u^{n+1} = (u+1)^n(2u) - u^{n+1}\,. \tag{5.2}$$

This implies that for arbitrary functions $g(u)$ we get

$$L\,g(u)|\mathcal{I}\rangle = u\big[2g(u+1) - g(u)\big]\,|\mathcal{I}\rangle\,. \tag{5.3}$$

In order to compute star products we need to invert (4.54) to find powers of u expressed in terms of powers of x, which multiply easily. For the first few cases we get

$$\begin{aligned} u &= \frac{x}{2}\,,\\ u^2 &= \Big(\frac{x}{2}\Big)^2 - \Big(\frac{x}{2}\Big)\,,\\ u^3 &= \Big(\frac{x}{2}\Big)^3 - 3\Big(\frac{x}{2}\Big)^2 + \Big(\frac{x}{2}\Big)\,,\\ u^4 &= \Big(\frac{x}{2}\Big)^4 - 6\Big(\frac{x}{2}\Big)^3 + 7\Big(\frac{x}{2}\Big)^2 - \Big(\frac{x}{2}\Big)\,. \end{aligned} \tag{5.4}$$

In general, we write:

$$u^n = Q_n\Big(\frac{x}{2}\Big) \equiv \sum_{k=1}^{n} q_{n,k}\Big(\frac{x}{2}\Big)^k\,, \tag{5.5}$$

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with coefficients $q_{n,k}$ that are defined by

$$q_{n,n} = 1\,, \quad q_{n,1} = (-1)^{n+1}\,, \tag{5.6}$$

and the recursion relation

$$q_{n,k} = q_{n-1,k-1} - k\, q_{n-1,k}\,, \quad k = 2, \ldots, n-1\,. \tag{5.7}$$

The star product $u^m\,|\mathcal{I}\rangle * u^n\,|\mathcal{I}\rangle$ can be evaluated by converting *one* of the factors explicitly to the x - basis. Indeed,

$$u^m\,|\mathcal{I}\rangle * u^n\,|\mathcal{I}\rangle = Q_m\Big(\frac{x}{2}\Big)|\mathcal{I}\rangle * Q_n\Big(\frac{x}{2}\Big)\,|\mathcal{I}\rangle = Q_n\Big(\frac{x}{2}\Big)\;Q_m\Big(\frac{x}{2}\Big)|\mathcal{I}\rangle = Q_n\Big(\frac{x}{2}\Big)\;u^m\,|\mathcal{I}\rangle\,. \tag{5.8}$$

We then have

$$u^m\,|\mathcal{I}\rangle * u^n\,|\mathcal{I}\rangle = \sum_{k=1}^{n} q_{n,k}\Big(\frac{x}{2}\Big)^k\; u^m|\mathcal{I}\rangle = \sum_{k=1}^{n} q_{n,k}\;(u+k)^m\,P_k(u)\;|\mathcal{I}\rangle\,. \tag{5.9}$$

Special cases are

$$\begin{aligned} u^m\,|\mathcal{I}\rangle * u\,|\mathcal{I}\rangle &= (u+1)^m u\,|\mathcal{I}\rangle\,, \\ u^m\,|\mathcal{I}\rangle * u^2\,|\mathcal{I}\rangle &= \big[(u+2)^m\,u(u+1) - (u+1)^m\,u\big]\,|\mathcal{I}\rangle\,. \end{aligned} \tag{5.10}$$

To check the accuracy of the $L^\star$ expansion, we examined the cases $s=1$ and $s=2$. For $s=1$ we obtained, at various levels,

$$\begin{aligned}
\ell^\star = 1: &\quad g_1(u) = 1 - 0.381966\,u\,,\\
\ell^\star = 2: &\quad g_1(u) = 1 - 0.422536u + 0.0658568\,u^2\,,\\
\ell^\star = 3: &\quad g_1(u) = 1 - 0.422745\,u + 0.0728081\,u^2 - 0.00518521\,u^3\,,\\
\ell^\star = 4: &\quad g_1(u) = 1 - 0.422788\,u + 0.0727478\,u^2 - 0.00491614u^3 - 0.000148237\,u^4\,,\\
\ell^\star = 5: &\quad g_1(u) = 1 - 0.422788\,u + 0.0728092\,u^2 - 0.00483623u^3 - 0.000328167\,u^4\,,\\
\ell^\star = 6: &\quad g_1(u) = 1 - 0.422785\,u + 0.0728153\,u^2 - 0.00484579u^3 - 0.000342449\,u^4\,.
\end{aligned} \tag{5.11}$$

(For levels 5 and 6 we have only written the coefficients up to u^4). We can compare the above with the exact solution $g_1(u)$ in (4.62), that expanded in powers of u gives

$$g_1(u) = 1 - 0.422784\,u + 0.072816\,u^2 - 0.004845\,u^3 - 0.000342\,u^4 + \cdots\,. \tag{5.12}$$

We see that the convergence of $L^\star$ expansion is excellent. The level six results are very accurate. We can also examine the coefficient $\gamma_2(1)$ in the descendant expansion (4.65) in the sliver conformal frame ($\alpha_2 = 2/3$). First note that its exact value is, from (4.68) and (4.62),

$$\gamma_2(1) = -\frac{1}{3} + \frac{4}{3} g_1(2) = \frac{1}{3}(8\zeta(3) - 9) \simeq 0.205485\,. \tag{5.13}$$

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The values of $\gamma_2(1)$ obtained from the level expansions in (5.11) are found by evaluation of $g_1(2)$. From $\ell^\star = 1$ to $\ell^\star = 6$ we find

$$L^\star - \text{expansion}: \quad \gamma_2(1) = -0.018576,\ 0.224474,\ 0.20568,\ 0.204953,\ 0.205375,\ 0.205428\,. \tag{5.14}$$

The convergence, again, is quite good. Had we truncated the exact solution, we would have instead obtained the values

$$\text{Truncated soln.}: \quad \gamma_2(1) = -0.127425,\ 0.260926,\ 0.209244,\ 0.201942,\ 0.206076,\ 0.205512\,. \tag{5.15}$$

Curiously, level expansion gets better partial results than the truncated expansion of the exact solution.

For $s = 2$ we get the following results

$$\begin{aligned}
\ell^\star = 1: &\quad g_2(u) = 1 - 0.438447\,u\,,\\
\ell^\star = 2: &\quad g_2(u) = 1 - 0.521156u - 0.102358\,u^2\,,\\
\ell^\star = 3: &\quad g_2(u) = 1 - 0.525210\,u + 0.123726\,u^2 - 0.0143123\,u^3\,,\\
\ell^\star = 4: &\quad g_2(u) = 1 - 0.524993\,u + 0.124554\,u^2 - 0.016306u^3 + 0.00101185\,u^4\,,\\
\ell^\star = 5: &\quad g_2(u) = 1 - 0.524998\,u + 0.124563\,u^2 - 0.0162718u^3 + 0.000954709\,u^4\,.
\end{aligned} \tag{5.16}$$

We clearly seem to have convergence. We can calculate the function g_2 at $u = 1$ and $u = 2$ and compare with (4.63). Using the level five solution we obtain

$$L^\star - \text{expansion}: \quad g_2(1) = 0.584248\,, \qquad g_2(2) = 0.333357\,, \tag{5.17}$$

in rather good agreement with (4.63). And then, for $\ell^\star = 1$ up to $\ell^\star = 5$ we find

$$\gamma_2(2) = 0.342329,\ 0.371804,\ 0.376304,\ 0.376401,\ 0.376402\,,$$

in nice agreement with the value recorded in (4.69).

We can even work in the limit $s \to \infty$. To order u^4, the level ten solution gives

$$g_\infty(u) = 1 - 0.693147\,u + 0.240226\,u^2 - 0.0555041u^3 + 0.00961812\,u^4 + \mathcal{O}(u^5)\,. \tag{5.18}$$

The expansion of the exact solution is

$$\begin{aligned}
g_\infty(u) = 2^{-u} &= 1 - (\ln 2)\,x + \frac{1}{2}(\ln 2)^2\,u^2 - \frac{1}{6}(\ln 2)^3\,u^3 + \frac{1}{24}(\ln 2)^4\,u^4 + \mathcal{O}(u^5)\\
&\simeq 1 - 0.693147\,u + 0.240227\,u^2 - 0.0555041\,u^3 + 0.00961813\,u^4 + \mathcal{O}(u^5)\,.
\end{aligned} \tag{5.19}$$

The agreement with the level-expansion solution is essentially perfect.

6. Examples and counterexamples

In this section we use examples to develop some intuition about special projectors. In section 6.1 we begin with a parameterized family of projectors which contains, for special values of the parameter, three special projectors. We demonstrate that only for the special

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projectors the state $|P_\infty\rangle$ coincides with the projector. All special projectors are annihilated by the derivation $K = \widetilde{L}^+$, since K and L^+ commute and K kills the identity. We provide an example in which we demonstrate that a projector that is not special fails to be annihilated by K.

In section 6.2 we discuss in detail the butterfly special projector. We give the explicit form for the frames $f_\alpha(\xi)$ that define the butterfly family $|P_\alpha\rangle$. We also discuss regularized butterflies. We briefly examine the special projectors that arise from $\mathcal{L}_0 = L_0 + (-1)^{m+1} L_{2m}$, $m \geq 1$.

6.1 A family of projectors

Consider the operator

$$\mathcal{L}_0 = L_0 + aL_2 + (a-1)L_4\,, \tag{6.1}$$

where a is a real constant whose possible values will define a family of surface states. The vector field associated with $\mathcal{L}_0$ is

$$v(\xi, a) = \xi + a\xi^3 + (a-1)\xi^5\,. \tag{6.2}$$

The coefficients have been adjusted so that $v(i,a) = 0$ — the vector field vanishes at the string midpoint. The conformal frame $z = f(\xi, a)$ can be obtained by solving the differential equation

$$\partial_\xi \ln f(\xi, a) = \frac{1}{v(\xi, a)} \quad \rightarrow \quad f(\xi, a) = \frac{\xi}{\left[1+\xi^2\right]^{\frac{1}{2(2-a)}}} \Big(1 + (a-1)\xi^2\Big)^{\frac{a-1}{2(2-a)}}\,. \tag{6.3}$$

We require the $f(\xi, a)$ to have no singularities for $|\xi| < 1$. This fixes $0 \leq a \leq 2$, but since the exponents become infinite for $a = 2$ we restrict our consideration to $0 \leq a < 2$. Note that in this range $f(\xi = i, a) = \infty$, so the surface states $\langle f(a)|$ are all projectors. We believe that these conformal frames obey conditions [**2**] and [**3**]. Indeed [**2a**] is obeyed by construction since $v(i,a) = 0$; [**2b**] seems unproblematic; [**3a**] is valid since $L^- = aK_2 + (a-1)K_4$ is a finite linear combination of K_n's. Finally, we expect [**3b**] to hold since $\widetilde{v}^+$, while not completely regular, has the same mild singularity as the butterfly.

We now ask for the values of a for which $\mathcal{L}_0$ and $\mathcal{L}_0^\star$ satisfy also [**1**] so that $\langle f(a)|$ are special projectors. A short computation gives:

$$[\,\mathcal{L}_0, \mathcal{L}_0^\star\,] = \big(6a^2 - 8a + 4\big) \left[2L_0 + \frac{2a(3a-2)}{6a^2-8a+4}\,(L_2 + L_{-2}) + \frac{4(a-1)}{6a^2-8a+4}(L_4 + L_{-4})\right]. \tag{6.4}$$

For the expression in brackets to be $\mathcal{L}_0 + \mathcal{L}_0^\star$ we need that the coefficients of $(L_4 + L_{-4})$ and $(L_2 + L_{-2})$ take the right values. For the former, comparing with (6.1), we get

$$\frac{4(a-1)}{6a^2-8a+4} = (a-1) \quad \rightarrow \quad a = 1, \quad \text{or} \quad 6a^2 - 8a + 4 = 4 \quad \rightarrow \quad a = 0,\; a = \frac{4}{3}\,. \tag{6.5}$$

One can readily verify that for the above three values of a the coefficients of $(L_2 + L_{-2})$ also work out. For $a = 1$ we recover the butterfly conformal frame and $s = 2$. For the

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other two values of a we get an algebra with $s = 4$. For $a = 0$ we have $\mathcal{L}_0 = L_0 - L_4$, a familiar projector [17, 20], to be further discussed in the next subsection. For $a = 4/3$ we get a new projector.

Let us now consider the conservation laws satisfied by the states $\langle f(a)|$. For all values $0 \le a < 2$, we have

$$\langle f(a)|\mathcal{L}_{-n} = 0\,, \qquad n \ge -1\,. \tag{6.6}$$

The above state that $\langle f(a)|$ is the vacuum in the conformal frame $f(\xi)$ — these conservations hold for any surface state. For special projectors, however, there is an additional conservation. We showed in section 3.3 that special projectors $\langle P|$ can be obtained as the $\gamma \to \infty$ limit of

$$\langle P| = \lim_{\gamma\to\infty} \langle \mathcal{I}|e^{-\gamma(\mathcal{L}_0+\mathcal{L}_0^\star)}\,. \tag{6.7}$$

This equation implies that

$$\langle P|(\mathcal{L}_0 + \mathcal{L}_0^\star) = 0\,. \tag{6.8}$$

We now demonstrate that in the family of projectors constructed above, only the special ones satisfy this extra conservation law. The requisite relation can be derived by considering the expansions

$$\mathcal{L}_{-n} = \oint \frac{d\xi}{2\pi i}\frac{(f(\xi,a))^{-n+1}}{f'(\xi,a)}\,T(\xi) = L_{-n} + \frac{1}{2}\,a\,(2+n)L_{-n+2} + \cdots \tag{6.9}$$

With the help of such relations for $\mathcal{L}_0, \mathcal{L}_{-2}$, and $\mathcal{L}_{-4}$ and (6.6) one can show that

$$0 = \langle f(a)| \Big(\mathcal{L}_0 + \mathcal{L}_0^\star + \frac{1}{6}a(a-1)(3a-4)\big[(2a-5)\,L_2 + (a-3)\,L_4\big] + \cdots\Big)\,. \tag{6.10}$$

We see that, apart from $(\mathcal{L}_0 + \mathcal{L}_0^\star)$, the additional terms shown above vanish only for the special projectors. One can show that for the special projectors the terms indicated by dots also vanish so that the states are annihilated by $\mathcal{L}_0 + \mathcal{L}_0^\star$. The above, however, is sufficient to conclude that (6.8) and (6.7) do not hold for general projectors. For the projectors in this family that are not special, we do not know what the state on the r.h.s. of (6.7) is.

As argued above, a special projector must also be annihilated by K. We want to show that this can fail to happen if the projector is not special. It is not easy to test this claim for the above family of states since their L^+ contains a finite number of operators and, consequently, K has an infinite number of operators. To build a testable example we begin with a derivation K that includes a finite number of Virasoro operators and construct the associated L^+ and surface state. We take

$$K = -3\,\frac{\pi}{2}\,K_3 = -3\,\frac{\pi}{2}\,(L_3 + L_{-3}) \quad \to \quad \widetilde{v^+} = -3\,\frac{\pi}{2}\left(\xi^4 + \frac{1}{\xi^2}\right), \tag{6.11}$$

where the constant of proportionality has been selected so that the dual vector corresponding to $\mathcal{L}_0 + \mathcal{L}_0^\star$ is well normalized

$$v^+(\xi) = -3\left(\xi^4 + \frac{1}{\xi^2}\right)\left(\tan^{-1}\xi + \tan^{-1}\left(\frac{1}{\xi}\right)\right) = \cdots + 2\xi + \cdots\,. \tag{6.12}$$

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One can check numerically that the operators $\mathcal{L}_0$ and $\mathcal{L}_0^\star$ do *not* satisfy the algebra [**1**]. The vector v corresponding to $\mathcal{L}_0$ can be read directly from the above

$$v(\xi) = \xi - \frac{18}{5}\,\xi^3 - \frac{18}{7}\,\xi^5 + \frac{2}{3}\,\xi^7 + \cdots \tag{6.13}$$

By integration of $f/f' = v$ we obtain the series expansion for the function f that defines the (non-special) projector $\langle f|$:

$$f(\xi) = \xi + \frac{9}{5}\,\xi^3 + \frac{963}{175}\,\xi^5 + \frac{29471}{1575}\,\xi^7 + \cdots \tag{6.14}$$

We then derive the conservation laws

$$\begin{aligned} 0 =& \langle f| \left(\; L_1 \; - \; \frac{9}{5}\,L_3 \; + \cdots \; \right), \\ 0 =& \langle f| \left(L_{-1} - \frac{27}{5}\,L_1 + \frac{288}{175}\,L_3 + \cdots \right), \\ 0 =& \langle f| \left(L_{-3} - 9\,L_{-1} + \frac{99}{5}\,L_1 + \frac{1471}{175}\,L_3 + \cdots \right). \end{aligned} \tag{6.15}$$

These equations imply that $0 \neq \langle f|K_3$, which is what we wanted to demonstrate. If the projector is not special $K = \widetilde{L}^+$ need not annihilate it.

6.2 The example of butterflies

The butterfly state $\langle \mathcal{B}| = \langle 0|e^{-\frac{1}{2}L_2}$ is the projector with local coordinate $f(\xi) = \xi/\sqrt{1+\xi^2}$. For the butterfly we have $v = f/f' = \xi + \xi^3$ and consequently

$$\mathcal{L}_0 = L_0 + L_2 \qquad \text{and} \qquad \mathcal{L}_0^\star = L_0 + L_{-2}\,. \tag{6.16}$$

In general,

$$\mathcal{L}_n = e^{\frac{1}{2}L_2}\,L_n\,e^{-\frac{1}{2}L_2}\,. \tag{6.17}$$

By construction we manifestly have the conservation laws:

$$\langle \mathcal{B}|\,\mathcal{L}_n = 0\,, \quad \text{for} \quad n \leq 1\,. \tag{6.18}$$

It follows from (6.17) that $\mathcal{L}_{-2k}$ with $k \geq -1$ consists of a finite linear combination of Virasoro operators in which the highest moded operator is L_2. Indeed, $\mathcal{L}_2 = L_2$, $\mathcal{L}_0 = L_0 + L_2$ and, quite interestingly,

$$\mathcal{L}_{-2} = L_{-2} + 2L_0 + L_2 = \mathcal{L}_0 + \mathcal{L}_0^\star\,. \tag{6.19}$$

It follows that the butterfly $\langle \mathcal{B}|$ is annihilated by both $\mathcal{L}_0$ and $\mathcal{L}_0^\star$. The butterfly is also annihilated by $\mathcal{L}_0 - \mathcal{L}_0^\star = L_2 - L_{-2} \equiv K_2$. Equation (6.19) implies the algebra [**1**] since

$$[\mathcal{L}_0\,, \mathcal{L}_{-2}\,] = 2\mathcal{L}_{-2} \quad \rightarrow \quad [\mathcal{L}_0\,, \mathcal{L}_0^\star\,] = 2\,(\mathcal{L}_0 + \mathcal{L}_0^\star)\,. \tag{6.20}$$

With $L \equiv \frac{1}{2}\mathcal{L}_0$ and $L^\star \equiv \frac{1}{2}\mathcal{L}_0^\star$ we have the canonically normalized algebra [**1**].

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The butterfly-based interpolating family $\langle P_\alpha|$ is defined by (3.1) using the butterfly L and $L^\star$. The conformal map $f_\alpha(\xi)$ corresponding to the state $\langle P_\alpha|$ is obtained using the general result (3.35). A computation gives

$$f_\alpha(\xi) = \frac{\xi\sqrt{1+\alpha+\alpha\,\xi^2}}{1+\alpha+(\alpha-1)\xi^2}\,. \tag{6.21}$$

For reference we also give the $\mathcal{L}_0$ operator as a function of α and the surface state:

$$\mathcal{L}_0(f_\alpha) = L_0 + \frac{\alpha-2}{\alpha+1}\,L_2 + \frac{2}{(1+\alpha)^2}\big(L_4 - L_6 + L_8 - L_{10} + \cdots\big)\,. \tag{6.22}$$

$$\langle P_\alpha| = \langle 0|e^{-A_\alpha}\,,\; A_\alpha = \frac{1}{2}\left(\frac{\alpha-2}{\alpha+1}\right)L_2 + \frac{1}{2}\frac{1}{(1+\alpha)^2}L_4 - \frac{1}{4}\left(\frac{2+\alpha}{(1+\alpha)^3}\right)L_6 + \cdots\,. \tag{6.23}$$

For $\alpha = 0$ we recover the $\mathcal{L}_0$ operator and the surface state expression of the identity state. For $\alpha \to \infty$ we recover the butterfly $\mathcal{L}_0$ and the Virasoro form of the state.

The family of states based on the butterfly provides a natural definition of a regulated butterfly as the state $\langle P_\alpha|$ for α large. The regulation is exact in the sense that the product of two such regulated butterflies is a regulated butterfly and, as the regulator is removed ($\alpha \to \infty$), we get the butterfly. The regulated butterflies of [17] multiply to give regulated butterflies only approximately. How do the states $\langle P_\alpha|$ look concretely? To answer this we examine $\langle P_2|$. For $\alpha = 2$, (6.21) gives

$$z = f_2(\xi) = \frac{\xi\sqrt{3+2\xi^2}}{3+\xi^2}\,. \tag{6.24}$$

The coordinate curve $f_2(e^{i\theta})$ is shown in figure 3(a). The map $f_2(\xi)$ does not extend as a one-to-one map from the ξ upper half-plane to the z upper-half plane. In order to get a one to one map, one must excise a small region bounded by dotted lines in the ξ-plane (see figure 3(b)). To help visualization, we show a ray r that barely touches the small region and its image in the z-plane. Since the surface cannot have a hole, there is a gluing instruction: points with equal imaginary coordinate on the dotted lines are to be identified. The identification is indicated in the figure with a short $\leftrightarrow$. Also noteworthy is that the part of the boundary beyond $|z| = \sqrt{2}$ in (a) appears as the vertical slit right above the excised region. After mapping the ξ upper-half plane to the unit w disk, the picture of the regulated butterfly is readily visualized (figure 3(c)). The regulation differs from that of [17] only by the presence of an excised region. We do not understand geometrically why the excision makes the regulation compatible with star multiplication. Such understanding may follow from a presentation in which the excised region takes a simple shape.

We examine briefly higher generalizations. Consider the conformal frames and associated projectors [17, 20]:

$$z = \xi\left(1+(-1)^{m+1}\xi^{2m}\right)^{-\frac{1}{2m}}\,, \qquad \langle\mathcal{P}^{(2m)}| = \langle 0|\exp\left[(-1)^m\frac{1}{2m}L_{2m}\right]\,. \tag{6.25}$$

The corresponding $\mathcal{L}_0$ operators satisfy [**1**] with $s = 2m$:

$$\mathcal{L}_0 = L_0 + (-1)^{m+1}L_{2m} \quad \to \quad [\,\mathcal{L}_0\,,\mathcal{L}_0^\star\,] = 2m(\mathcal{L}_0 + \mathcal{L}_0^\star)\,. \tag{6.26}$$

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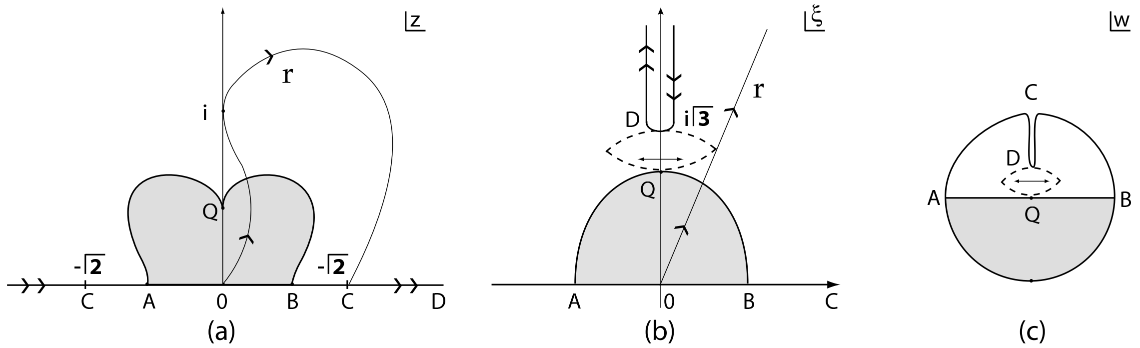

Figure 3: The surface state $\langle P_2|$ in the butterfly family. The coordinate disk in the z UHP is shown in (a). The map from ξ to z can be extended to a full map of upper half planes if, in the ξ plane, we cut out the region bounded by dotted lines and glue the resulting boundary points horizontally. In w coordinates the result is recognized as a regulated butterfly.

Eq. (6.19) readily generalizes to

$$\mathcal{L}_{-2m} = (-)^{m+1}(\mathcal{L}_0 + \mathcal{L}_0^\star), \tag{6.27}$$

and therefore [$\underline{\mathbf{1}}$] follows from the Virasoro algebra in the frame of the projector:

$$[\mathcal{L}_0, \mathcal{L}_{-2m}] = 2m\mathcal{L}_{-2m} \quad \rightarrow \quad [\,\mathcal{L}_0\,, \mathcal{L}_0^\star\,] = 2m(\mathcal{L}_0 + \mathcal{L}_0^\star)\,. \tag{6.28}$$

The states $\langle P_\alpha^m|$ associated with these special projectors have conformal frames

$$f_\alpha^m(\xi) = \frac{\xi\big[1+\alpha+\alpha(-1)^{m+1}\,\xi^{2m}\big]^{\frac{1}{2m}}}{\big[1+\alpha+\alpha(-1)^{m+1}\,\xi^{2m}\big]^{\frac{1}{m}} - \xi^2}\,. \tag{6.29}$$

The corresponding $\mathcal{L}_0(f_\alpha^m)$ operator takes the form

$$\begin{aligned}\mathcal{L}_0(f_\alpha^m) = \;& L_0 - \frac{2}{(1+\alpha)^{\frac{1}{m}}}\,L_2 + \frac{2}{(1+\alpha)^{\frac{2}{m}}}\,L_4 - \;\ldots\; + (-1)^{m-1}\,\frac{2}{(1+\alpha)^{1-\frac{1}{m}}}\,L_{2m-2} \\ & + (-1)^{m+1}\,\frac{\alpha-2}{\alpha+1}\,L_{2m} + \frac{2}{m}(-1)^m\sum_{k=1}^{m}\frac{(m-k)\alpha-m}{(1+\alpha)^{1+\frac{k}{m}}}L_{2m+2k} + \;\ldots\end{aligned} \tag{6.30}$$

The states $\langle \mathcal{P}^{(2m)}|$ are an infinite family of special projectors.

7. Conformal frames for $[\mathcal{L}_0, \mathcal{L}_0^\star] = s(\mathcal{L}_0 + \mathcal{L}_0^\star)$

We have seen that condition [$\underline{\mathbf{1}}$], which states that the operators $\mathcal{L}_0$ and $\mathcal{L}_0^\star$ form the algebra

$$[\,\mathcal{L}_0\,, \mathcal{L}_0^\star\,] = s(\mathcal{L}_0 + \mathcal{L}_0^\star)\,, \tag{7.1}$$

is necessary for the kinetic operator $\mathcal{L}_0$ to have simple action on string fields of the form $f(L^+)|\mathcal{I}\rangle$. Since $\mathcal{L}_0^\star$ is readily obtained by BPZ conjugation of $\mathcal{L}_0$, it is simply the choice

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of $\mathcal{L}_0$ that determines if the algebra (7.1) holds. Moreover, the choice of $\mathcal{L}_0$ is the choice of a conformal map $z = f(\xi)$, as indicated in (2.1).

It is the purpose of this section to determine the class of *special conformal frames*, defined as the functions $f(\xi)$ that result in operators $\mathcal{L}_0$ and $\mathcal{L}_0^\star$ that satisfy (7.1). Note that we do not impose a priori conditions [**2**] and [**3**] so, as we shall see in concrete examples, special conformal frames need not be projectors. The set of special conformal frames is characterized by continuous parameters for each integer s, but the number of special projectors may be finite for each s.

The analysis that follows has several parts. We derive the relevant constraint in section 7.1. We analyze the constraint and relate it to the Riemann-Hilbert problem in section 7.2. Solutions are presented in section 7.3, where we look at examples of special projectors for $s = 1$, $s = 2$, and $s = 3$. In section 7.4 we discuss a general pattern that seems to emerge from the examples: the operator $\mathcal{L}_{-s}$ is always related by a "generalized" duality to the operator L^+. This duality makes [**1**] a consequence of the Virasoro commutator $[\mathcal{L}_0, \mathcal{L}_{-s}] = s\mathcal{L}_{-s}$. We also construct an interesting infinite family of special projectors.

7.1 Deriving the constraint

Our strategy is to show that the algebra (7.1) implies a second-order differential equation for $f(\xi)$. Happily, the differential equation can be integrated twice to give a tractable condition on $f(\xi)$. In fact, the condition constrains the values of f on the unit circle $|\xi| = 1$.

We begin the work by considering the vector fields associated with the operators $\mathcal{L}_0$ and $\mathcal{L}_0^\star$. Equation (2.19) furnishes the vector $v(\xi)$ associated with $\mathcal{L}_0$:

$$v(\xi) = \frac{1}{(\ln f(\xi))'} \tag{7.2}$$

Let us now determine the vector $v^\star(\xi)$ associated with $\mathcal{L}_0^\star$. Using (2.24) and letting primes denote derivatives with respect to the argument

$$v^\star(\xi) = -\xi^2\, v(-1/\xi) = -\xi^2\, \frac{f(-1/\xi)}{f'(-1/\xi)} = \frac{-1}{\partial_\xi \ln f(-1/\xi)} = \frac{-1}{(\ln f \circ I(\xi))'}, \text{ with } I(\xi) = -\frac{1}{\xi}\,. \tag{7.3}$$

Given (2.20), we will realize the algebra (7.1) if we have

$$[\, v(\xi)\,, v^\star(\xi)\,] = -s(v(\xi) + v^\star(\xi))\,. \tag{7.4}$$

This condition gives the equation

$$v\partial_\xi v^\star - v^\star\partial_\xi v = -s(v + v^\star)\,. \tag{7.5}$$

This involves first derivatives of the vectors, so second derivatives of $f(\xi)$. A little algebra using (7.2) and (7.3) gives

$$\frac{\big(\ln\circ f\circ I\big)''}{\big(\ln\circ f\circ I\big)'} - \frac{\big(\ln\circ f\big)''}{\big(\ln\circ f\big)'} = s\Big(\big(\ln\circ f\big)' - \big(\ln\circ f\circ I\big)'\Big)\,. \tag{7.6}$$

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After integration and some rearrangement we find

$$\frac{\big(\ln\circ f\circ I\big)'(f\circ I)^s}{\big(\ln\circ f\big)' f^s} = -(-1)^s\, C_1\,, \tag{7.7}$$

where C_1 is a constant and the sign factor has been introduced for convenience. This can be integrated once again to give the final equation, with two constants of integration:

$$\left[f\Big(-\frac{1}{\xi}\Big)\right]^s + (-1)^s C_1 \left[f(\xi)\right]^s = (-1)^s C_2\,. \tag{7.8}$$

Since the function $f(\xi)$ is only guaranteed to exist for $|\xi| \le 1$, this equation is only a condition on the values of the function $f(\xi)$ on the circle $|\xi| = 1$. As before, to emphasize this point we use $t = e^{i\theta}$ to denote points on the unit circle. Noting that $1/t^* = t$, we write

$$\left[f(-t^*)\right]^s + (-1)^s C_1 \left[f(t)\right]^s = (-1)^s C_2\,. \tag{7.9}$$

As we will see below, in general C_1 and C_2 cannot be constants over the circle. The solutions $f(\xi)$ that we find (that include the sliver and butterfly frames) are singular on a set of points on the unit disk. These points break the circle in a set of intervals; C_1 and C_2 need only be constants over each of those intervals.

7.2 Solving the constraint

We begin our analysis of (7.9) by imposing some conditions on the functions $f(\xi)$. Since $f(\xi)$ maps the real boundary of the ξ half-disk to the real axis on the z-plane, we must have

$$f(\xi^*) = [f(\xi)]^*\,, \tag{7.10}$$

where $*$ denotes complex conjugation. For simplicity, we further restrict our analysis to twist invariant surface states,

$$f(-\xi) = -f(\xi)\,. \tag{7.11}$$

Equations (7.11) and (7.10) imply that

$$f(-\xi^*) = -[f(\xi)]^*\,. \tag{7.12}$$

This equation has a clear geometrical meaning: points that are reflections about the imaginary ξ-axis map to points that are reflections about the imaginary z-axis. We conclude that the coordinate curve for the map f (the image under f of the curve $|\xi| = 1, \mathrm{Im}(\xi) \ge 0$) is symmetric under reflection about the imaginary z-axis. Note also that $[f(i)]^* = f(i^*) = f(-i) = -f(i)$, so $f(i)$, if finite, must be purely imaginary.

The above conditions on $f(\xi)$ show that equation (7.9) must be handled with care. For example, letting $t \to -t$ makes the left-hand side go to $(-1)^s$ times itself, so for s odd C_2 cannot be a constant over the whole circle. Since $f(-t) = -f(t)$, we need only consider (7.9) for the half-circle $\mathrm{Re}(t) \ge 0$: if f is determined there it is known over the rest of the circle. Because of the complex conjugation relation (7.10) we can restrict ourselves

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further to $t = e^{i\theta}$ with $0 \leq \theta \leq \frac{\pi}{2}$. We look for solutions in which this quarter circle is split into N intervals by points θ_i where the function becomes singular

$$0 = \theta_0 < \theta_1 < \ldots < \theta_{N-1} < \theta_N = \frac{\pi}{2}\,. \tag{7.13}$$

Using (7.10) for the first term in (7.9) and cancelling a common factor of $(-1)^s$ we obtain the conditions

$$\begin{aligned}
&\Big[\big(f(t)\big)^s\Big]^* + C_1(k)\,\big(f(t)\big)^s = C_2(k)\,, \\
&\qquad\qquad t = e^{i\theta}, \qquad \theta_{k-1} \leq \theta \leq \theta_k, \qquad k = 1, \ldots, N \\
&f(-t) = -f(t)\,, \qquad\qquad f(t^*) = (f(t))^*\,.
\end{aligned} \tag{7.14}$$

As indicated by their argument k, both C_1 and C_2 can take different (constant) values over the intervals. We now recognize the main condition above as a case of the general Riemann-Hilbert problem for a disk. In this problem one looks for an analytic function $\Phi(\xi)$ on the disk $|\xi| < 1$. On the boundary $|\xi| = 1$ of the disk the function and its complex conjugate satisfy a relation of the form

$$(\alpha(t) + i\beta(t))\Phi(t) + (\alpha(t) - i\beta(t))(\Phi(t))^* = \gamma(t)\,, \tag{7.15}$$

for real functions $\alpha(t), \beta(t)$ and $\gamma(t)$, with $\alpha^2 + \beta^2 \neq 0$ [42]. As we show next, over each interval the constants C_1 and C_2 must take values consistent with the structure of (7.15). To streamline the notation we use

$$F(\xi) \equiv \big[f(\xi)\big]^s\,, \tag{7.16}$$

and we rewrite the main condition as

$$[F(t)]^* + C_1(k)\,F(t) = C_2(k)\,, \tag{7.17}$$

or, more briefly, as

$$F^* + C_1\,F = C_2\,. \tag{7.18}$$

Taking the complex conjugate and dividing by C_1^* we get

$$F^* + \frac{1}{C_1^*}\,F = \frac{C_2^*}{C_1^*}\,. \tag{7.19}$$

Taking the difference of the last two equations we find

$$\left(C_1 - \frac{1}{C_1^*}\right)\,F = C_2 - \frac{C_2^*}{C_1^*}\,. \tag{7.20}$$

A constant $F(t)$ is not a satisfactory solution because it implies that $f(t)$ is unchanged as t varies. We therefore need $C_1 C_1^* = 1$, or

$$C_1(k) = e^{2i\alpha_k}\,, \quad \alpha_k \text{ real}\,. \tag{7.21}$$

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In this case the left-hand sides of equations (7.18) and (7.19) are identical, and the equality of the right-hand sides gives $C_2 = C_1 C_2^*$. This implies that C_2 is given by

$$C_2(k) = 2r_k\, e^{i\alpha_k}\,, \quad r_k \text{ real}\,. \tag{7.22}$$

Back in (7.18) we get

$$F^* + e^{2i\alpha_k}\, F = 2r_k\, e^{i\alpha_k}\,, \tag{7.23}$$

or, equivalently,

$$e^{i\alpha_k}\, F + e^{-i\alpha_k} F^* = 2r_k\;. \tag{7.24}$$

We now see that this equation is precisely of the Riemann-Hilbert form (7.15). For our problem, the functions α, β, and γ are piecewise constants.[12] It follows from (7.24) that

$$\mathrm{Re}\big[\, F\, e^{i\alpha_k}\,\big] = r_k\,. \tag{7.25}$$

We have therefore shown that

$$\mathrm{Re}\Big[\big(f(t)\big)^s\; e^{i\alpha_k}\Big] = r_k\,, \quad t = e^{i\theta}, \;\; \theta_{k-1} \leq \theta \leq \theta_k\,, \;\; k = 1, \ldots, N\,. \tag{7.26}$$

The condition is very simple: over each interval the function $u = (f(t))^s$ must lie on a straight line ℓ_k in the u-plane. The line is specified by the (largely) arbitrary constants α_k and r_k. The value of α_k is the rotation angle about $u = 0$ that makes ℓ_k vertical. The value of r_k is the value of the real coordinate for that vertical line. A solution is specified by fixing some $N \geq 1$, the angles $\theta_1, \ldots \theta_{N-1}$ that fix the intervals, and constants α_k and r_k for each interval.

The above prescription provides the coordinate curve that defines the function $f(\xi)$ but does not provide the function itself. There are two ways to obtain this function. In the first way, we solve the corresponding Riemann-Hilbert problem, which, in general, expresses the solution in terms of fairly involved Cauchy integrals. We will not attempt to do so here, although we have verified that this procedure works as expected for the case of the sliver. In the second way, the function $f(\xi)$ is determined by the conformal map that takes the upper-half ξ disk to the coordinate disk.

While the prescription indicated below (7.26) to build a coordinate curve provides a solution, the solution is sometimes formal and not always produces an $f(\xi)$ with operators $\mathcal{L}_0$ and $\mathcal{L}_0^\star$ that satisfy (7.1). There are conditions on the ranges of allowed angles, certainly on the interval that contains $\xi = i$. We leave the discussion of the matter incomplete and proceed to illustrate with examples some large classes of solutions that we have checked are not formal.

[12] In order to obtain an analytic function that is not singular anywhere on the boundary of the disk, one must have continuous functions α, β, and γ, or more precisely, the functions must satisfy Holder conditions [42]. Our functions $f(\xi)$ can have singularities on the boundary of the disk.

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7.3 Explicit solutions for special frames

We now turn to describe some concrete examples of special frames and special projectors. We illustrate the existence of special frames that are not special projectors by certain deformations of the sliver. We will not attempt to classify all special projectors. It appears that for a fixed s positive and integer there is a finite number of special projectors. For $s = 1$ we only find one, the sliver. For $s = 2$, besides the familiar butterfly, we find an interesting new projector, the moth. For $s = 3$ we discuss two examples. All the cases that we study exhibit a common pattern: the operator $\mathcal{L}_{-s}$ in the frame of the special projector plays a crucial role.

7.3.1 The case $s = 1$

The condition here is simply that the coordinate curve is made of piecewise linear functions. We only need to describe the curve to the right of the imaginary axis because the rest of the curve is obtained by reflection. The simplest and most familiar solution is provided by the vertical line $\text{Re}(z) = \pi/4$ and $\text{Im}(z) \geq 0$. The coordinate disk maps to the strip $-\frac{\pi}{4} \leq \text{Re}(z) \leq \frac{\pi}{4}$ and $\text{Im}(z) \geq 0$. We recognize this as the rectangular strip of the sliver map:

$$f(\xi) = \tan^{-1}(\xi)\,, \qquad (s = 1)\,. \tag{7.27}$$

The corresponding $\mathcal{L}_0$ operator is

$$\mathcal{L}_0 = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{4k^2 - 1}\, L_{2k} = L_0 + \frac{2}{3}L_2 - \frac{2}{15}L_4 + \frac{2}{35}L_6 - \cdots \quad . \tag{7.28}$$

Another solution is provided by the horizontal line $\text{Im}(z) = \frac{\pi}{4}$. In this case the coordinate disk maps to the strip $0 \leq \text{Im}(z) \leq \frac{\pi}{4}$, One can check that the mapping function is

$$f(\xi) = \tanh^{-1}(\xi)\,, \qquad (s = 1)\,. \tag{7.29}$$

The $\mathcal{L}_0$ operator for this map is obtained from the sliver $\mathcal{L}_0$ by reversing the sign of each Virasoro operator whose mode number is twice odd:

$$\mathcal{L}_0 = L_0 - \sum_{k=1}^{\infty} \frac{2}{4k^2 - 1}\, L_{2k} = L_0 - \frac{2}{3}L_2 - \frac{2}{15}L_4 - \frac{2}{35}L_6 - \cdots \quad . \tag{7.30}$$

Since $f(\xi = i)$ is finite the surface state is not a projector. The vector $v = (\xi^2 - 1)\tanh^{-1}\xi$ corresponding to $\mathcal{L}_0$ does not vanish at $\xi = i$.

We now discuss a family of special frames that interpolate between the sliver and the "horizontal sliver" (7.29). Consider the case where the coordinate curve in the z plane is the isosceles triangle with vertices A and B on the real line and Q on the imaginary axis. The angle at Q is $q\pi$ with $0 \leq q \leq 1$. As we traverse the edges of the triangle in the counterclockwise direction the turning angles at A and B are both equal to α, where

$$\frac{\alpha}{\pi} = \frac{1}{2}(q + 1)\,. \tag{7.31}$$

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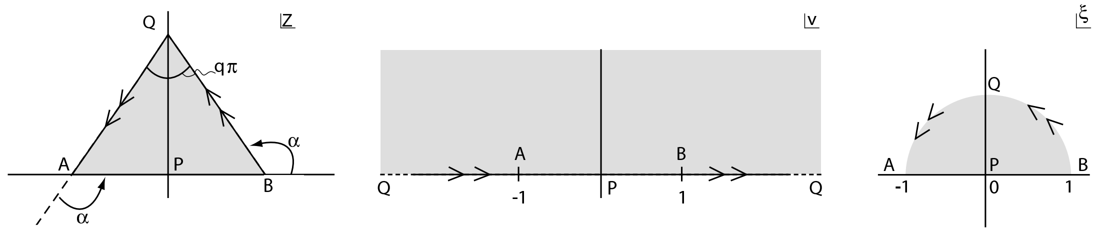

Figure 4: Conformal maps for the q-deformations of the sliver. The angle at the apex of the triangle (leftmost figure) is πq. The sliver is recovered for $q = 0$.

The map $z = f(\xi)$ is constructed in two steps. In the first one we build a map from the interior of the triangle to the upper half v-plane with the edges of the triangle going to the real line and the base of the triangle mapping to the real segment between $v = -1$ and $v = 1$. The Schwarz-Christoffel differential equation is

$$\frac{dz}{dv} = A(v-1)^{-\frac{\alpha}{\pi}}(v+1)^{-\frac{\alpha}{\pi}}\,, \tag{7.32}$$

where the magnitude of A can be adjusted since it just fixes the arbitrary scale in the z plane. We thus take $|A|$=1 and choose the phase of A such that the equation becomes

$$\frac{dz}{dv} = \frac{1}{(1-v^2)^{\frac{\alpha}{\pi}}}\,. \tag{7.33}$$

We thus write

$$z = \int_0^v \frac{du}{(1-u^2)^{(q+1)/2}}\,. \tag{7.34}$$

The integral is readily expressed in terms of hypergeometric functions:

$$z = v \,\cdot\, {}_2F_1\left[\,\frac{1}{2},\,\frac{1}{2}(q+1),\,\frac{3}{2},\,v^2\right]. \tag{7.35}$$

This is the desired map from the upper-half v plane to the triangle. The map from the coordinate half-disk ξ to the upper-half v plane is

$$v = \frac{2\xi}{1+\xi^2}\,. \tag{7.36}$$

This is, in fact, the map that defines the nothing state [17]. The full map we are looking for is simply the composition of the two maps above:

$$z(\xi) = \frac{2\xi}{1+\xi^2}\,\cdot\,{}_2F_1\left[\,\frac{1}{2},\,\frac{1}{2}(q+1),\,\frac{3}{2},\,\frac{4\xi^2}{(1+\xi^2)^2}\right]. \tag{7.37}$$

Although complicated, series expansions are readily found:

$$z(\xi) = \xi + \left(-\frac{1}{3}+\frac{2}{3}q\right)\xi^3 + \left(\frac{1}{5}-\frac{2}{5}q+\frac{2}{5}q^2\right)\xi^5 + \cdots \tag{7.38}$$

$$\mathcal{L}_0(q) = L_0 + \left(\frac{2}{3}-\frac{4}{3}q\right)L_2 + \left(-\frac{2}{15}-\frac{16}{15}q+\frac{16}{15}q^2\right)L_4 + \cdots\,, \tag{7.39}$$

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and the surface state is

$$\langle \Sigma_q | = \langle 0 | \exp(M) \,, \tag{7.40}$$

where

$$M = -\left(\frac{1}{3} - \frac{2}{3}q\right) L_2 + \left(\frac{1}{30} + \frac{4}{15}q - \frac{4}{15}q^2\right) L_4 + \left(-\frac{11}{1890} + \frac{29}{315}q - \frac{76}{315}q^2 + \frac{152}{945}q^3\right) L_6 + \cdots \,. \tag{7.41}$$

We have tested numerically that the operators $\mathcal{L}_0(q)$ and $\mathcal{L}_0^\star(q)$ satisfy the algebra (7.1) with $s = 1$ in the range $q \in [0,1]$. The q-deformed sliver states $\langle \Sigma_q |$ are not projectors because $f(\xi = i)$ is finite for $q \neq 0$. In fact, near the midpoint one has

$$f(\xi) = f_0 + f_1 (\xi - i)^q + \cdots \,, \tag{7.42}$$

with q-dependent constants f_0 and f_1. For $q \neq 0$, f_0 is finite, and near the midpoint the vector field $v(\xi)$ associated with $\mathcal{L}_0(q)$ takes the form

$$v(\xi) = \frac{f(\xi)}{f'(\xi)} = \frac{f_0}{q f_1} (\xi - i)^{1-q} + \cdots \tag{7.43}$$

The vector $v(\xi)$ has a fractional power zero at the string midpoint. So will the BPZ dual vector $v^\star$, the vector $v^+ = v + v^\star$ associated with L^+, and the vector $\widetilde{v^+} = v^+ \epsilon$ associated with K. We believe that K fails to annihilate the identity. The reason is the following: since the q-deformed slivers are not projectors they are not special projectors. Thus they must fail to satisfy at least one of the three conditions listed in the introduction. They satisfy condition [**1**] and condition [**2a**]. The projector property does not require [**3a**]. So, the q-deformed slivers fail to satisfy [**2b**] or [**3b**], or both. Property [**2b**] does not strike us as too delicate, so we feel the culprit is [**3b**] — the failure of K to kill the identity.

For similar reasons we suspect that the coordinate curve cannot have corners anywhere. If it did, $v(\xi)$ would have fractional power zeros and those may result in a K that does not kill the identity. If this is indeed the case, we have the conclusion that the sliver provides the unique special projector for $s = 1$.

7.3.2 The case $s = 2$

For $s = 2$ we have found two special projectors: the butterfly and a new projector, the moth.

The butterfly is a familiar example. Indeed, with

$$f(\xi) = \frac{\xi}{\sqrt{1 + \xi^2}} \,, \tag{7.44}$$

we verify that for $\xi = e^{i\theta}$

$$(f(\xi))^2 = \frac{e^{2i\theta}}{1 + e^{2i\theta}} = \frac{e^{i\theta}}{2\cos\theta} = \frac{1}{2} + \frac{i}{2} \tan\theta \,. \tag{7.45}$$

Since this is a line (in fact vertical), the butterfly map satisfies the condition (7.26) for $s = 2$. Note that, up to an irrelevant scale, the butterfly coordinate curve is the square root of the sliver coordinate curve (see figure 5(a)).

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A natural generalization of the sliver suggests itself:

$$z = f(\xi) = \left[\tan^{-1} \xi^2\right]^{1/2} . \tag{7.46}$$

The square of this function, $\tan^{-1} \xi^2$, maps the circle $|\xi| = 1$ to vertical lines, just like the sliver function $\tan^{-1} \xi$ does. Thus the constraint (7.26) is satisfied with $s = 2$. On the other hand, we do not get a projector since $f(\xi = i)$ is finite.

The following function, however, works all the way:

$$z = f(\xi) = \left[\frac{1}{2} \ln\left(\frac{1+\xi^2}{1-\xi^2}\right)\right]^{1/2} = \left(\tanh^{-1} \xi^2\right)^{1/2} = \xi + \frac{1}{6}\,\xi^5 + \frac{31}{360}\,\xi^9 + \cdots . \tag{7.47}$$

One can readily check that $(f(\xi))^2$ maps the unit circle to a horizontal line, thus satisfying the constraint (7.26) with $s = 2$. Moreover $f(\xi = i)$ is infinite. So, we have a special projector (see figure 5(b)), which we shall call the moth. A quick computation gives an $\mathcal{L}_0$ reminiscent of the sliver's (7.28):

$$\mathcal{L}_0 = L_0 + \sum_{k=1}^{\infty} \frac{2(-1)^k}{4k^2-1}\, L_{4k} . \tag{7.48}$$

Numerically tests indicate that, as expected, the algebra (7.1) is satisfied. With v denoting the vector associated with $\mathcal{L}_0$ one finds that:

$$v^+ = \frac{1}{2}(v + v^\star) = \frac{1-\xi^4}{2\xi}\,\left(\tanh^{-1}(\xi^2) - \tanh^{-1}(1/\xi^2)\right) . \tag{7.49}$$

We can use (3.35) to calculate the maps $f_\alpha(\xi)$ that define the $\langle P_\alpha|$ states of the moth family. Including a convenient overall normalization constant, we find

$$f_\alpha(\xi) = \frac{\sqrt{1+\alpha}}{2}\left[\left(\frac{1+\xi^2}{1-\xi^2}\right)^{\frac{2}{1+\alpha}} - 1\right]^{1/2} . \tag{7.50}$$

This family of states was encountered before in [17], where it appeared as the most general family of (twist invariant) surface states annihilated by the derivation $K_2 = L_2 - L_{-2}$.[13] Interestingly, a quick calculation shows that $\mathcal{L}_{-2} = -K_2$. The Virasoro algebra in the moth frame gives $[\mathcal{L}_0, K_2] = 2K_2$, whose BPZ conjugate is $[\mathcal{L}_0^\star, K_2] = -2K_2$. It follows that $[L^+, K_2] = 0$, which explains why the whole family, constructed as an exponential of L^+ acting on the identity, is killed by K_2. The states $|P_\alpha\rangle$ are also annihilated by a second derivation, the operator $K = \widetilde{L}^+$, which is an infinite linear combination of K_{2n+1} operators and thus clearly different from K_2. Note also that $[K_2, K] = \widetilde{[K_2, L^+]} = 0$. For $\alpha = 1$ we find $f_1(\xi) = \xi/\sqrt{1-\xi^2}$, which is the butterfly map with the "wrong" sign inside the square root. The corresponding state is

$$|P_1\rangle = e^{\frac{1}{2}L_{-2}}\,|0\rangle . \tag{7.51}$$

[13] Comparing with eq. (9.10) of [17], we find that the parameter μ in (9.10) is related to the parameter α in (7.50) as $\mu = -1/(1+\alpha)$. In the present context $\alpha \geq 0$, hence $-1 \leq \mu \leq 0$. In the context of [17], $\mu > 0$ is legitimate, and one recognizes e.g. the butterfly state for $\mu = 1/2$ and the nothing state for $\mu = 1$.

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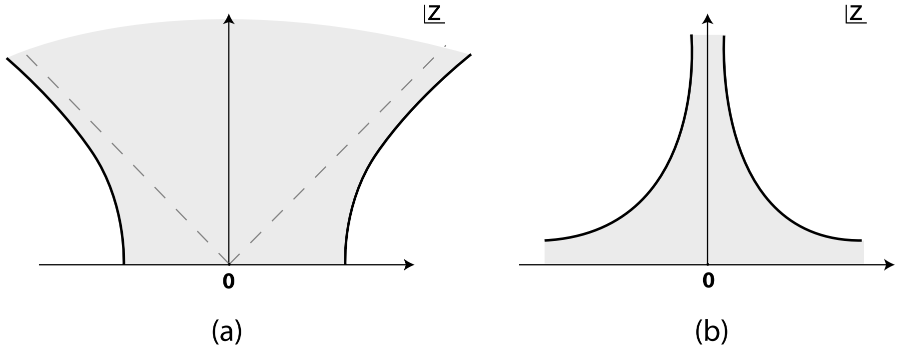

Figure 5: Coordinate curves for $s = 2$ special projectors. (a) The coordinate curve for the butterfly. (b) The coordinate curve for the moth (7.47).

This wrong-sign butterfly has been studied in [20]. According to [20] (eq. (57)), the product of two wrong-sign butterflies is the surface state associated with the function (7.50) with $\alpha = 2$, in nice agreement with $|P_1\rangle * |P_1\rangle = |P_2\rangle$. The multiplication of an infinite number of wrong-sign butterfly states $|P_1\rangle$ gives the moth projector $|P_\infty\rangle$ described by (7.47).

7.3.3 The case $s = 3$

For $s = 3$ we construct explicitly two special projectors.

The generalization (7.46) of the sliver, that did not work for $s = 2$, works for $s = 3$:

$$z = f(\xi) = \left[\tan^{-1}\xi^3\right]^{1/3} = \xi - \frac{1}{9}\,\xi^7 + \frac{22}{405}\,\xi^{13} + \cdots\,. \tag{7.52}$$

Indeed, condition (7.26) is satisfied with $s = 3$ and $f(\xi = i) = \infty$. In fact $f(\xi = e^{i\pi/6}) = \infty$. This example corresponds to a realization of (7.26) with $N = 2$ with $\theta_1 = \pi/6$. The coordinate curve for this projector is shown in figure 6(a). With v denoting the vector associated with $\mathcal{L}_0$ one finds that:

$$v^+ = \frac{1}{3}(v + v^\star) = \frac{1+\xi^6}{3\xi^2}\,\left(\tan^{-1}(\xi^3) + \tan^{-1}(1/\xi^3)\right). \tag{7.53}$$

A quick calculation shows that $\mathcal{L}_{-3} = K_3 = L_3 + L_{-3}$. We can repeat the argument given for the moth to show that K_3 commutes with L^+ and thus annihilates the whole family P_α based on this projector. The family will also be killed by $K = \widetilde{L^+} \neq K_3$.

This suggests looking for a projector for which K *is* proportional to $\mathcal{L}_{-3}$. Such projector is a different kind of generalization of the sliver, which satisfies $\mathcal{L}_{-1} \sim K$. Moreover, $K \sim K_1$, so the vector field $(1+\xi^2)$ associated with the sliver $\mathcal{L}_{-1}$ vanishes for $\xi = \pm i$. For the $s = 3$ projector we take $\mathcal{L}_{-3} = K_3 + aK_1$ and adjust the constant a so that the vector associated with $\mathcal{L}_{-3}$ has zeroes only at $\xi = \pm i$. This gives $a = 3$, so we get

$$\mathcal{L}_{-3} = K_3 + 3K_1 \qquad \rightarrow \qquad v_{\mathcal{L}_{-3}} = \frac{(1+\xi^2)^3}{\xi^2}\,. \tag{7.54}$$

We also have

$$\mathcal{L}_{-3} = \oint \frac{d\xi}{2\pi i}\,\frac{3}{(f^3)'}\,T(\xi) \quad \rightarrow \quad v_{\mathcal{L}_{-3}} = \frac{3}{(f^3)'}\,. \tag{7.55}$$

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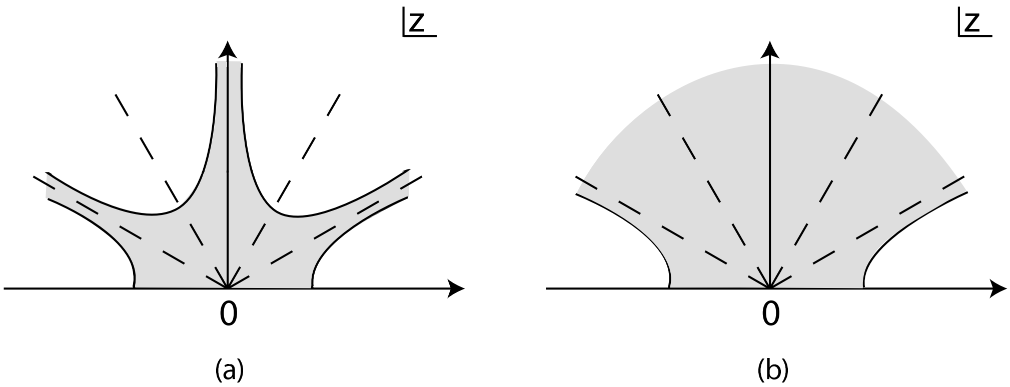

Figure 6: Coordinate curves for $s = 3$ special projectors. (a) The coordinate curve for the (7.52) projector. (b) The coordinate curve for the projector (7.56).

The last two equations give a differential equation for f that is readily integrated:

$$f(\xi) = (3/8)^{1/3} \left[\tan^{-1}\xi + \frac{\xi^3 - \xi}{(1+\xi^2)^2}\right]^{1/3} = \xi - \frac{3}{5}\,\xi^3 + \frac{87}{175}\,\xi^5 + \cdots . \tag{7.56}$$

We now verify that K is indeed proportional to $\mathcal{L}_{-3}$. We first use f to calculate the vector v associated with $\mathcal{L}_0$ and then form

$$v^+(\xi) = \frac{1}{3}(v + v^\star) = \frac{(1+\xi^2)^3}{8\xi^2}\Big(\tan^{-1}\xi + \tan^{-1}(1/\xi)\Big)\,, \tag{7.57}$$

so that

$$v_K = \widetilde{v^+}(\xi) = v^+(\xi)\,\epsilon(\xi) = \frac{\pi}{16}\,\frac{(1+\xi^2)^3}{\xi^2} = \frac{\pi}{16}\,v_{\mathcal{L}_{-3}} \quad \to \quad K = \frac{\pi}{16}\,\mathcal{L}_{-3}\,, \tag{7.58}$$

as desired. The relation $K \sim \mathcal{L}_{-3}$ explains why (7.1) holds: the Virasoro algebra commutator $[\mathcal{L}_0, K] = 3K$ upon dualization gives $[\mathcal{L}_0, L^+] = 3L^+$. This is equivalent to (7.1) with $s = 3$. The coordinate curve for the special projector (7.56) is shown in figure figure 6(b). Note that the coordinate curve is a sub-curve of that for the projector (7.52), shown in figure 6(a). The coordinate disks differ, of course.

Other $s = 3$ special projectors are likely to exist. For example, a projector whose coordinate curve (in the region $\mathrm{Re}\,z > 0$) is the cubic root of a horizontal line may exist, according to (7.26). It would be the $s = 3$ analog of (7.47).

7.4 Generalized duality

In all the examples of special projectors that we have considered the operator $\mathcal{L}_{-s}$ is interesting in some way. We have seen that for each even s there is a (higher) butterfly projector (6.25) for which $\mathcal{L}_{-s} \sim L^+$. The algebra [**1**] then follows from the Virasoro commutator $[\mathcal{L}_0, \mathcal{L}_{-s}] = s\mathcal{L}_{-s}$. For each odd s there is a "dual" construction where $\mathcal{L}_{-s} \sim K = \widetilde{L^+}$. The $s = 1$ and $s = 3$ constructions give the sliver and the projector (7.56), respectively. This construction generalizes to all odd s, as we will discuss at the end of the present section. The algebra [**1**] now follows from $[\mathcal{L}_0, \mathcal{L}_{-s}] = s\mathcal{L}_{-s}$ and duality.

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The two remaining examples, the $s = 2$ moth (7.47) and the $s = 3$ projector (7.52), follow a somewhat different pattern. In these cases $\mathcal{L}_{-s}$ is very simple — it is proportional, respectively, to the derivations K_2 and K_3 — but has no apparent connection with the fundamental objects L^+ and K. To establish this connection we generalize the notion of duality.

We introduce the function $\epsilon_\alpha(t)$, with $0 \leq \alpha < \pi$, defined for $t = e^{i\theta}$ by

$$\epsilon_\alpha(e^{i\theta}) = \begin{cases} 1\,, & \text{if} \quad \alpha - \pi < \theta \leq \alpha\,; \\ -1\,, & \text{if} \quad \alpha < \theta \leq \alpha + \pi. \end{cases} \tag{7.59}$$

In other words, we bisect the unit circle at an angle α and define ϵ_α to take values ± 1 in the two halves. The function $\epsilon(t)$ that we have used so far is $\epsilon_{\pi/2}(t)$. Two functions on a circle will be said to be dual to each other if one is equal to the other times the product of a finite number of ϵ_α functions, with different values of α. Clearly, squaring a product of ϵ_α's gives the constant function 1 on the circle.

Consider now the function that appears multiplicatively in the v^+ vector (7.49) of the moth. We claim that

$$\tanh^{-1}(t^2) - \tanh^{-1}\left(1/t^2\right) = -i\,\frac{\pi}{2}\,\epsilon_{\pi/2}(t)\,\epsilon_0(t)\,. \tag{7.60}$$

Indeed, for any $t = e^{i\theta}$ on the circle the left-hand side is equal to $\pm i\pi/2$. The signs alternate over angular intervals of ninety degrees, with value $+i\pi/2$ for $\theta \in [0, \pi/2]$. The product on the right-hand side reproduces this behavior. Recalling, additionally, that $\mathcal{L}_{-2} = -K_2$, we have

$$v_{\mathcal{L}_{-2}}(t) = \frac{1 - \xi^4}{\xi}\,. \tag{7.61}$$

It now follows from (7.49) and the last two equations that for the moth

$$v^+(t) = -i\,\frac{\pi}{4}\,\epsilon_{\pi/2}(t)\,\epsilon_0(t)\;v_{\mathcal{L}_{-2}}(t)\,, \tag{7.62}$$

showing that after all, $v_{\mathcal{L}_{-2}}$ and v^+, as well as $\mathcal{L}_{-2}$ and L^+, are related by duality. We can now dualize the $\mathcal{L}_{-2}$ in the commutator $[\mathcal{L}_0, \mathcal{L}_{-2}] = 2\mathcal{L}_{-2}$ to obtain the relation $[\mathcal{L}_0, L^+] = 2L^+$. This dualization is allowed following the logic of (2.60), which demands that the product of the underlying vectors vanishes at the discontinuities of the dualizing function. This holds, since the vector corresponding to $\mathcal{L}_0$ vanishes for $\xi = \pm i$ and ± 1.

Dualizing (7.62) by further multiplication by $\epsilon_{\pi/2}(t)$ we get

$$v_K(t) = i\,\frac{\pi}{4}\,\epsilon_0(t)\,v_{K_2}(t)\,. \tag{7.63}$$

This means that K is proportional to the dual of K_2, with duality flipping the sign of the vector on the upper half circle $\Im t > 0$. The vector v_{K_2} vanishes at the discontinuities $t = \pm 1$ of $\epsilon_0(t)$, so we can conclude that $[K, K_2]$ is dual to the commutator $[K, K]$ and must therefore vanish.

The $s = 3$ projector (7.52) can be understood similarly. The vector v^+ given in (7.53) contains the multiplicative factor $(\tan^{-1}(t^3) + \tan^{-1}(1/t^3))$ that is equal to $\pm\pi/2$ over six

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alternating intervals of the unit circle, with plus sign for $\theta \in [-\pi/6, \pi/6]$. This time one finds the interesting relation

$$v^+ = \frac{\pi}{6}\,\epsilon_{\pi/2}\,\epsilon_{\pi/6}\,\epsilon_{5\pi/6}\;v_{\mathcal{L}_{-3}}\,. \tag{7.64}$$

The vector $v_{\mathcal{L}_{-3}} = v_{K_3} = \xi^4 + 1/\xi^2$ vanishes at the discontinuities of the dualizing function so, once again, $[\underline{\mathbf{1}}]$ follows by dualization of the $[\mathcal{L}_0, \mathcal{L}_{-3}]$ commutator.

In summary, for all special projectors known so far, the operator $\mathcal{L}_{-s}$ is related to L^+ (and to K) by a generalized duality. This duality relation, which might be generic, "explains" why these conformal frames are special.

Finally, we would like to briefly describe an interesting infinite family of special projectors, which contains as special cases several examples that we have already discussed. For every integer s we look for the conformal frame f in which the vector field associated with $\mathcal{L}_{-s}$ is

$$v_{\mathcal{L}_{-s}} \equiv \frac{s}{(f^s)'} = \frac{(1+\xi^2)^s}{\xi^{s-1}}\,. \tag{7.65}$$

Note that $v_{\mathcal{L}_{-s}}$ is BPZ even (odd) for s even (odd). Integrating (7.65), we find

$$f(\xi) = \xi\left({}_2F_1\left[\frac{s}{2}, s\,; 1+\frac{s}{2}\,; -\xi^2\right]\right)^{1/s}\,. \tag{7.66}$$

These conformal frames are projectors for all real $s \geq 1$: $f_s(\pm i) = \infty$.

For $s = 1$ the hypergeometric function simplifies and we recover the sliver. Curiously, for $s = -1$ we find the map of the identity. For $s = 3$ we recover the projector (7.56), as we should given (7.54). Using a standard hypergeometric identity we can rewrite (7.66) as

$$f(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}\left({}_2F_1\left[\frac{s}{2}, 1-\frac{s}{2}\,; 1+\frac{s}{2}\,; \frac{\xi^2}{\xi^2+1}\right]\right)^{1/s}\,. \tag{7.67}$$

This presentation makes it manifest that for s even the hypergeometric function truncates to a finite polynomial of its argument. For $s = 2$ we recognize the butterfly map. For $s = 4$ we recover the special projector (6.3) with $a = 4/3$. We claim that for each even s, the operator $\mathcal{L}_{-s}$ is proportional to L^+, while for each odd s it is proportional to K. We simply quote the result, obtained using hypergeometric identities,

$$\frac{\Gamma(s/2+1)\Gamma(s/2)}{\Gamma(s+1)}\;\mathcal{L}_{-s} = \begin{cases} L^+ & \text{for } s \text{ even}\,, \\ K & \text{for } s \text{ odd}\,. \end{cases} \tag{7.68}$$

This result implies that the algebra $[\underline{\mathbf{1}}]$ holds and the projectors are special. It would be interesting to investigate if (7.66) defines special projectors even for non-integer s. It seems clear that we have just begun to understand the rich algebraic structure of special projectors.

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8. Concluding remarks

Since a summary of our results was given in the introductory section, we limit ourselves here to point out some questions that remain open.

At a technical level, it would be interesting to have a complete classification of special projectors. Perhaps the ideas of generalized duality explained in section 7.4 will turn out to be useful. Explicit forms for the function $f(\xi)$ that defines the projectors are in general missing. We have not determined if there are special projectors for non-integer s. A complete analysis will require understanding what are the conditions on the vector v associated with $\mathcal{L}_0$ that ensure that properties [**3a**] and [**3b**] hold. It also became clear in our work that the framework of conservation laws in the operator formalism requires generalization to deal with vector fields that, having certain singularities on the unit circle, do not define analytic functions over the rest of the complex plane.

It remains somewhat surprising that there is a notion of a special projector. For a special projector one finds a family of states, built using a rather simple prescription, that interpolate from the identity to the projector. One wonders if a related, perhaps more complicated construction, exists for arbitrary projectors. We have noted that for arbitrary projectors the corresponding $|P_\alpha\rangle$ states multiply as expected but $|P_\infty\rangle$ is not the projector one starts with.

In order to make the techniques discussed here applicable to the physical ghost-number one equation of motion one may generalize the abelian algebra $\mathcal{A}_f$ to include the action of ghost oscillators. This should suffice to construct the string field corresponding to the tachyon vacuum. An additional extension of $\mathcal{A}_f$ to include matter oscillators seems necessary to produce solutions that describe D-brane solitons, Wilson lines, and the time-dependent decay of D-branes.

In the process of extending the results to the ghost and matter sectors we will be able to find out if there is something truly special about the sliver that allowed Schnabl to find a solution, or if all special projectors are on the same footing. We think this question is an important one, since it can eventually help simplify the solution and understand better its universal features. Such understanding is likely to point out ways to obtain new and perhaps unexpected solutions of open string field theory.

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A. Lie algebra and Lie group relations

We consider the nonabelian Lie algebra with two generators

$$[\, L \,,\, L^\star \,] = L + L^\star \,, \tag{A.1}$$

and the corresponding adjoint representation

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \,, \quad L^\star = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \,. \tag{A.2}$$

As matrices we have the following relations

$$LL = L \,, \quad L^\star L^\star = -L^\star \,, \quad LL^\star = L^\star \,, \quad L^\star L = -L \,. \tag{A.3}$$

Note also the relation

$$(L + L^\star)^2 = 0 \,. \tag{A.4}$$

In this representation group elements are given by

$$e^{\alpha L + \beta L^\star} = 1 + \frac{e^{\alpha - \beta} - 1}{\alpha - \beta} \, (\alpha L + \beta L^\star) \,. \tag{A.5}$$

Particular useful cases are

$$x^L = 1 + (x-1) L \,, \quad y^{L^\star} = 1 + \left(1 - \frac{1}{y} \right) L^\star \,. \tag{A.6}$$

The CBH formula for this group can be derived by comparing group elements in the adjoint representation. We find that

$$e^{\alpha L} \, e^{\beta L^\star} = e^{\beta' L^\star} \, e^{\alpha' L} = e^{\tilde\alpha L + \tilde\beta L^\star} \tag{A.7}$$

determines $(\tilde\alpha, \tilde\beta)$ in terms of (α, β) or in terms of (α', β'):

$$\begin{aligned} \tilde\alpha &= \frac{\alpha - \beta}{e^{-\beta} - e^{-\alpha}} \, (1 - e^{-\alpha}) = \frac{\alpha' - \beta'}{e^{\alpha'} - e^{\beta'}} \, (e^{\alpha'} - 1) \,, \\ \tilde\beta &= \frac{\alpha - \beta}{e^{-\beta} - e^{-\alpha}} \, (1 - e^{-\beta}) = \frac{\alpha' - \beta'}{e^{\alpha'} - e^{\beta'}} \, (e^{\beta'} - 1) \,. \end{aligned} \tag{A.8}$$

We also have the following inverse relations:

$$\begin{aligned} e^{\alpha} &= \left(1 - \frac{\tilde\alpha}{\tilde\beta} \right)^{-1} \left[1 - \frac{\tilde\alpha}{\tilde\beta} \, e^{\tilde\alpha - \tilde\beta} \right] \,, \qquad & e^{\beta} &= \left(1 - \frac{\tilde\alpha}{\tilde\beta} \right)^{-1} \left[e^{\tilde\beta - \tilde\alpha} - \frac{\tilde\alpha}{\tilde\beta} \right] \,, \\ e^{\alpha'} &= \left(1 - \frac{\tilde\alpha}{\tilde\beta} \right) \left[1 - \frac{\tilde\alpha}{\tilde\beta} \, e^{\tilde\beta - \tilde\alpha} \right]^{-1} \,, \qquad & e^{\beta'} &= \left(1 - \frac{\tilde\alpha}{\tilde\beta} \right) \left[e^{\tilde\alpha - \tilde\beta} - \frac{\tilde\alpha}{\tilde\beta} \right]^{-1} \,. \end{aligned} \tag{A.9}$$

We could also give the relations $(\alpha, \beta) \leftrightarrow (\alpha', \beta')$. But it is more useful to write them as Schnabl, who gives the first one of the following:

$$\begin{aligned} x^L \, y^{L^\star} &= \left(\frac{y}{x + y - xy} \right)^{L^\star} \left(\frac{x}{x + y - xy} \right)^{L} \,, \\ y^{L^\star} \, x^L &= \left(\frac{x + y - 1}{y} \right)^{L} \left(\frac{x + y - 1}{x} \right)^{L^\star} \,. \end{aligned} \tag{A.10}$$

Useful corollaries are

$$x^{L-L^\star} = \left(\frac{2}{1+x^2}\right)^{L^\star}\left(\frac{2x^2}{1+x^2}\right)^{L} = \left(\frac{1+x^2}{2}\right)^{L}\left(\frac{1+x^2}{2x^2}\right)^{L^\star} \tag{A.11}$$

as well as the fairly redundant but helpful identities

$$\begin{aligned} x^{L^\star} x^L &= e^{\left(1-\frac{1}{x}\right)(L+L^\star)} = \left(2-\frac{1}{x}\right)^{L}\left(2-\frac{1}{x}\right)^{L^\star}, \\ x^L\, x^{L^\star} &= e^{(x-1)(L+L^\star)} = \left(\frac{1}{2-x}\right)^{L^\star}\left(\frac{1}{2-x}\right)^{L}, \\ e^{x(L+L^\star)} &= \left(\frac{1}{1-x}\right)^{L^\star}\left(\frac{1}{1-x}\right)^{L} = (x+1)^L\,(x+1)^{L^\star}. \end{aligned} \tag{A.12}$$

References

[1] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [hep-th/0511286].

[2] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory*, *JHEP* **04** (2006) 055 [hep-th/0603159].

[3] E. Fuchs and M. Kroyter, *On the validity of the solution of string field theory*, *JHEP* **05** (2006) 006 [hep-th/0603195].

[4] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum*, *JHEP* **02** (2007) 096 [hep-th/0606142].

[5] A. Sen, *Non-BPS states and branes in string theory*, hep-th/9904207; *Tachyon dynamics in open string theory*, *Int. J. Mod. Phys.* **A 20** (2005) 5513 [hep-th/0410103].

[6] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys.* **B 268** (1986) 253.

[7] L. Rastelli and B. Zwiebach, *Tachyon potentials, star products and universality*, *JHEP* **09** (2001) 038 [hep-th/0006240].

[8] V.A. Kostelecky and R. Potting, *Analytical construction of a nonperturbative vacuum for the open bosonic string*, *Phys. Rev.* **D 63** (2001) 046007 [hep-th/0008252].

[9] L. Rastelli, A. Sen and B. Zwiebach, *Classical solutions in string field theory around the tachyon vacuum*, *Adv. Theor. Math. Phys.* **5** (2002) 393 [hep-th/0102112].

[10] L. Rastelli, A. Sen and B. Zwiebach, *Boundary CFT construction of D-branes in vacuum string field theory*, *JHEP* **11** (2001) 045 [hep-th/0105168].

[11] L. Rastelli, A. Sen and B. Zwiebach, *Vacuum string field theory*, hep-th/0106010; D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Ghost structure and closed strings in vacuum string field theory*, *Adv. Theor. Math. Phys.* **6** (2003) 403 [hep-th/0111129].

[12] L. Rastelli, A. Sen and B. Zwiebach, *Half strings, projectors and multiple D-branes in vacuum string field theory*, *JHEP* **11** (2001) 035 [hep-th/0105058].

[13] D.J. Gross and W. Taylor, *Split string field theory. I*, *JHEP* **08** (2001) 009 [hep-th/0105059]; *Split string field theory. II*, *JHEP* **08** (2001) 010 [hep-th/0106036].

JHEP01(2008)020

[14] H. Hata and T. Kawano, *Open string states around a classical solution in vacuum string field theory*, *JHEP* **11** (2001) 038 [hep-th/0108150].

[15] L. Rastelli, A. Sen and B. Zwiebach, *Star algebra spectroscopy*, *JHEP* **03** (2002) 029 [hep-th/0111281].

[16] M. Schnabl, *Wedge states in string field theory*, *JHEP* **01** (2003) 004 [hep-th/0201095].

[17] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Star algebra projectors*, *JHEP* **04** (2002) 060 [hep-th/0202151].

[18] I. Bars, *Map of Witten's * to Moyal's **, *Phys. Lett.* **B 517** (2001) 436 [hep-th/0106157]; *MSFT: Moyal star formulation of string field theory*, hep-th/0211238.

[19] M.R. Douglas, H. Liu, G.W. Moore and B. Zwiebach, *Open string star as a continuous Moyal product*, *JHEP* **04** (2002) 022 [hep-th/0202087].

[20] M. Schnabl, *Anomalous reparametrizations and butterfly states in string field theory*, *Nucl. Phys.* **B 649** (2003) 101 [hep-th/0202139].

[21] E. Fuchs, M. Kroyter and A. Marcus, *Squeezed state projectors in string field theory*, *JHEP* **09** (2002) 022 [hep-th/0207001].

[22] L. Bonora, C. Maccaferri, D. Mamone and M. Salizzoni, *Topics in string field theory*, hep-th/0304270.

[23] L. Bonora, C. Maccaferri and P. Prester, *Dressed sliver solutions in vacuum string field theory*, *JHEP* **01** (2004) 038 [hep-th/0311198].

[24] Y. Okawa, *Some exact computations on the twisted butterfly state in string field theory*, *JHEP* **01** (2004) 066 [hep-th/0310264].

[25] Y. Okawa, *Solving Witten's string field theory using the butterfly state*, *Phys. Rev.* **D 69** (2004) 086001 [hep-th/0311115].

[26] E. Fuchs and M. Kroyter, *On surface states and star-subalgebras in string field theory*, *JHEP* **10** (2004) 004 [hep-th/0409020].

[27] W. Taylor and B. Zwiebach, *D-branes, tachyons and string field theory*, hep-th/0311017.

[28] L. Rastelli, *String field theory*, hep-th/0509129.

[29] W. Taylor, *String field theory*, hep-th/0605202.

[30] E. Fuchs and M. Kroyter, *Schnabl's $\mathcal{L}_0$ operator in the continuous basis*, *JHEP* **10** (2006) 067 [hep-th/0605254].

[31] D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, *Patterns in open string field theory solutions*, *JHEP* **03** (2002) 003 [hep-th/0201159].

[32] J. Kluson, *Exact solutions of open bosonic string field theory*, *JHEP* **04** (2002) 043 [hep-th/0202045].

[33] I. Kishimoto and K. Ohmori, *CFT description of identity string field: toward derivation of the VSFT action*, *JHEP* **05** (2002) 036 [hep-th/0112169].

[34] T. Takahashi and S. Tanimoto, *Marginal and scalar solutions in cubic open string field theory*, *JHEP* **03** (2002) 033 [hep-th/0202133].

JHEP01(2008)020

[35] V.A. Kostelecky and S. Samuel, *On a nonperturbative vacuum for the open bosonic string*, *Nucl. Phys.* **B 336** (1990) 263.

[36] A. Sen and B. Zwiebach, *Tachyon condensation in string field theory*, *JHEP* **03** (2000) 002 [hep-th/9912249].

[37] N. Moeller and W. Taylor, *Level truncation and the tachyon in open bosonic string field theory*, *Nucl. Phys.* **B 583** (2000) 105 [hep-th/0002237].

[38] D. Gaiotto and L. Rastelli, *Experimental string field theory*, *JHEP* **08** (2003) 048 [hep-th/0211012].

[39] E. Witten, *Interacting field theory of open superstrings*, *Nucl. Phys.* **B 276** (1986) 291.

[40] A. LeClair, M.E. Peskin and C.R. Preitschopf, *String field theory on the conformal plane.* 1*. Kinematical principles*, *Nucl. Phys.* **B 317** (1989) 411; *String field theory on the conformal plane.* 2*. Generalized gluing*, *Nucl. Phys.* **B 317** (1989) 464.

[41] I. Ellwood, B. Feng, Y.-H. He and N. Moeller, *The identity string field and the tachyon vacuum*, *JHEP* **07** (2001) 016 [hep-th/0105024].

[42] N.I. Muskhelishvili, *Singular integral equations*, P. Noordhoff N.V., Groningen, Holland (1953).

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